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DUAL ALGEBRAS AND A-MEASURES.

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Let A be an arbitrary function algebra. The main subject of our investigation are properties of the spectrum of A^{**}.

Motivation:

- A - measures problem
- the problem for which $G \subset \mathbb{C}^n$ the algebra $H^\infty(G)$ is a dual algebra
- the application of dual algebras in functional calculus for bounded operators in Hilbert spaces
- connections with the Corona problem

As an application of our main result we have obtained:

- a general positive solution for A - measures problem
- the duality of $H^\infty(G)$ algebra for some classes of bounded domains $G \subset \mathbb{C}^n$

Definition 1. A is a function algebra on a compact set X iff $A \subset C(X)$, A contains constants and separates the points of X

Let $\phi, \psi \in \sigma(A)$

$$\phi \sim \psi \overset{\text{df}}{\iff} \|\phi - \psi\| < 2$$

Definition 2. The equivalence classes in the above equivalence relation are called *Gleason parts* of A.

We assume $\sigma(A) = X$.

Denote by $M(X)$ the Banach space of all complex Borel regular measures on X equiped by the total variation norm. A set $\mathcal{M} \subset M(X) = C(X)^*$ is a band if it is a closed subspace and $\mu \in \mathcal{M}, \nu \ll |\mu| \implies \nu \in \mathcal{M}$. Every measure $\mu \in M(X)$ has a unique Lebesque decomposition $\mu = \mu_{\mathcal{M}} + \mu_s$ where $\mu_{\mathcal{M}} \in \mathcal{M}$ and μ_s is singular to each measure in $\mathcal{M}$. We say that $\mathcal{M}$ is a reducing band (with respect to A) if $\mu \in A^\perp \implies \mu_{\mathcal{M}} \in A^\perp$. A measure ν is a representing measure for $x \in X = \sigma(A)$ if $f(x) = \int f\, d\nu$ for $f \in A$ For a subset G of X we denote by $\mathcal{M}_G$ the band generated by G i.e. the smallest band containing all measures representing for points in G. If G is a Gleason part then $\mathcal{M}_G$ is a reducing band.

Since $C(X)^{**} := (C(X)^*)^*$ is a commutative, symmetric C^* algebra, by Gelfand-Naimark theorem there exist a hyperstonean compact space Y such that $C(X)^{**} = M(X)^* \approx C(Y)$ in the sense of isometric isomorphism. Each $f \in C(X)$ can be treated as a functional on $M(X)$ and consequently as an element of $C(Y)$ by the formula

$$\langle f, \mu \rangle = \int f\, d\mu \quad \text{for} \quad \mu \in M(X).$$

For $\mu \in M(X)$ there is a unique measure $\tilde{\mu} \in M(Y) := C(Y)^*$ such that $\langle F, \mu \rangle = \int F\, d\tilde{\mu}$ for all $F \in C(Y)$.

Theorem 3. *If G is a Gleason part of A then the weak-star closure $\overline{G}^{ws}$ of G in Y is a closed-open subset of Y. Moreover*

$$Y \setminus \overline{G}^{ws} = \overline{X \setminus G}^{ws}, \quad (\overline{\mathcal{M}_G}^{ws})^s = \overline{(\mathcal{M}_G^s)}^{ws}, \quad \overline{\mathcal{M}_G}^{ws} = M(\overline{G}^{ws}),$$

and $\overline{\mathcal{M}_G}^{ws}$ is a reducing band for A^{**}.

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Corollary 4. *There exists a characteristic function $F_0 \in A^{**}$ vanishing exactly on $Y \setminus \overline{G}^{ws}$ and the projection associated with the decomposition $M(Y) = \overline{\mathcal{M}_G}^{ws} + \overline{\mathcal{M}_G^s}^{ws}$ is exactly the multiplication by F_0.*

Corollary 5. *If G is a Gleason part of a function algebra A, $x \in G$ and μ_x is any its representing measure, then μ_x is concentrated on the weak-star closure of G.*

We assume that G is a Gleason part of A and denote by $H^\infty(\mathcal{M}_G)$ - the weak-star closure of A in $\mathcal{M}_G^*$. By the definition of $H^\infty(\mathcal{M}_G)$, the values of its every element are uniquely defined on each $x \in G$.

Proposition 6. *$H^\infty(\mathcal{M}_G)$ is isometrically isomorphic to $A^{**}/_{\mathcal{M}_G^\perp \cap A^{**}}$*

Corollary 7. *G is a subset of the spectrum of $H^\infty(\mathcal{M}_G)$.*

Theorem 8. *If G is a Gleason part of A, then $H^\infty(\mathcal{M}_G)$ satisfies the domination condition:*

$$\|f\| = \sup_{x \in G} |f(x)| \qquad \textit{for any} \quad f \in H^\infty(\mathcal{M}_G).$$

Proposition 9. *The band $\mathcal{M}_G$ is equal to the norm closed linear span of all representing measures for points in G, taken in the quotient space $M(X)/A^\perp$.*

For $f \in H^\infty(\mathcal{M}_G)$ and $z \in G$ we can define $f(z)$ as the value of f on a representing measure ν_z for z. By the weak-star density of A in $H^\infty(\mathcal{M}_G)$, the value $f(z)$ does not depend on the choice of representing measure. So the elements of $H^\infty(\mathcal{M}_G)$ can be regarded as functions on G.

Proposition 10. *If G is a bounded domain in $\mathbb{C}^n$ and $f \in H^\infty(\mathcal{M}_G)$ then the defined above mapping $z \to f(z)$ is a bounded analytic function of $z \in G$.*

Proposition 11. *If G is a star-shaped domain in $\mathbb{C}^n$ such that $\overline{G}$ is the spectrum of $A(G)$, then the algebras $H^\infty(G)$ and $H^\infty(\mathcal{M}_G)$ are isometrically isomorphic. Hence $H^\infty(G)$ is a dual algebra.*

Open problem. Is $\sigma(A^{**}) = Y/_{(A^{**})^\perp}$, where Y is the spectrum of $C(X)^{**}$?

Consequences. If the above open problem would have a positive solution, then the Corona problem would have a positive solution for the case when $H^\infty(G)$ and $H^\infty(\mathcal{M}_G)$ are isometrically isomorphic.

Assume $Q = \bigcup_\alpha G_\alpha$, where for each α, G_α is a Gleason part of A.

Definition 12. We say that a measure $\mu \in M(X)$ is an *A-measure* (or analytic measure, or a Henkin measure) with respect to the the set Q if $\int u_n \, d\mu \to 0$ whenever $\{u_n\}_{n=1}^\infty \subset A$ is a bounded sequence converging to 0 pointwise on Q.

A-measures problem for the algebra A at the points of Q. Does the absolute continuity of a measure μ on X with respect to some representing measure of a point $x \in Q$ imply that μ is an A-measure?

Another formulation. Is any measure which is absolutely continuous with respect to a positive A-measure, itself an A-measure?

Theorem 13. *If A is a function algebra on X and $Q \subset X$ is equal to a countable union of its Gleason parts, then A-measures problem for the algebra A at the points of Q has a positive solution.*

Corollary 14. *The A-measures problem at the points of $Q = G$ for $A(G)$ has a positive solution if G is either a strictly pseudoconvex set in $\mathbb{C}^n$, or a Carthesian product of a finite number of such domains.*

This includes polydiscs, polydomains (products of bounded plane domains), but also products of balls with polydiscs.

Theorem 15. *A-measures problem for the algebra $A = H^\infty(G)$ at all points of a countable union Q of its arbitrary Gleason parts has a positive solution. In particular, if G is a star-shaped domain in $\mathbb{C}^n$ such that $\overline{G}$ is the spectrum of $A(G)$, then A-measures problem for $H^\infty(G)$ at all points of G has a positive solution.*

Before our results, A-measures problem was solved positively by advanced complex analysis methods for two special cases:

- by Cole and Range for X being the closure of a strictly pseudoconvex bounded domain Q in $\mathbf{C}^n$ with C^2 boundary, and A being the algebra of all complex continuous functions on X which are holomorphic on its interior Q
- by Bekken and Bui Doan Khanh in the case of the cartesian product of compact planar sets for two classes of algebras - for algebras of continuous functions which are holomorphic on the interior and for algebras generated by rational functions with singularities off X

Both above cases are covered by our results.

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