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SET OF OPERATORS WITH 0 IN THE CLOSURE OF THE NUMERICAL RANGE

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1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space. Denote by $\mathcal{B}(\mathscr{H})$ the Banach algebra of all bounded linear operators on $\mathscr{H}$ and by $\mathcal{S}_{\mathscr{H}} = \{x \in \mathscr{H};\ \|x\| = 1\}$ the unit sphere of $\mathscr{H}$. The numerical range of $A \in \mathcal{B}(\mathscr{H})$ is

$$W(A) = \{\langle Ax, x\rangle;\ x \in \mathcal{S}_{\mathscr{H}}\}.$$

It is obvious that $W(A)$ is a non-empty subset of $\mathbb{C}$ which is contained in the disk $\{z \in \mathbb{C};\ |z| \leq \|A\|\}$. If $\dim(\mathscr{H}) < \infty$, then $W(A)$ is a closed set. However, if $\mathscr{H}$ is not finite-dimensional, then the numerical range is not closed, in general. For instance, the numerical range of the backward shift on ℓ^2 is the open unit disk. One among the basic properties of the numerical range is its convexity.

It is well-known that the spectrum of A is contained in the closure of the numerical range, i.e., $\sigma(A) \subseteq \overline{W(A)}$. Because of the convexity of the numerical range, one actually has $\mathrm{conv}(\sigma(A)) \subseteq \overline{W(A)}$, where $\mathrm{conv}(\cdot)$ denotes the convex hull of a set. For some operators the opposite inclusion holds, as well — normal operators have this property, for instance — but for a general operator the above inclusion can be proper. However Hildebrandt [2] has proved the following theorem.

Theorem 1. *For every $A \in \mathcal{B}(\mathscr{H})$, one has* $\mathrm{conv}\big(\sigma(A)\big) = \bigcap_{\substack{S\in\mathcal{B}(\mathscr{H})\\ \text{invertible}}} \overline{W(SAS^{-1})}$.

2. Main result

Let $\mathscr{W}_{\{0\}} = \{A \in \mathcal{B}(\mathscr{H});\ 0 \in \overline{W(A)}\}$. It is obvious that this is a proper non-empty subset of $\mathcal{B}(\mathscr{H})$. It is not hard to see that it is closed in the norm topology. As the following proposition shows, set $\mathscr{W}_{\{0\}}$ is quite large.

Proposition 2. *If $\dim(\mathscr{H}) \geq k + 1$, then $\mathscr{W}_{\{0\}}$ is k-transitive, that is, for every linearly independent vectors $x_1, \ldots, x_k \in \mathscr{H}$ and for every set of k vectors $\{y_1, \ldots, y_k\} \subseteq \mathscr{H}$ there exists an operator $A \in \mathscr{W}_{\{0\}}$ such that $Ax_i = y_i$ $(i = 1, \ldots, k)$.*

Proof. Note that the set of all singular operators is contained in $\mathscr{W}_{\{0\}}$. It is not hard to see that actually the set of singular operators is k-transitive. Indeed, let $x_1, \ldots, x_k \in \mathscr{H}$ be linearly independent and let $\{y_1, \ldots, y_k\} \subseteq \mathscr{H}$ be an arbitrary set of k vectors. Since $\dim(\mathscr{H}) \geq k+1$ there exists a vector $e \in \mathcal{S}_{\mathscr{H}}$ such that $x_i \perp e$ for every $i = 1, \ldots, k$. Because of linear independence of vectors $e, x_1, \ldots, x_k$ there exists $A \in \mathcal{B}(\mathscr{H})$ such that $Ae = 0$ and $Ax_i = y_i$ $(i = 1, \ldots, k)$. □

Set $\mathscr{W}_{\{0\}}$ is not closed under addition or multiplication. For instance, let $P \neq 0, I$ be an orthogonal projection. Then $P, I - P$, and the involution $U = 2P - I$ are in $\mathscr{W}_{\{0\}}$, but $P + (I - P) = I$ and $U^2 = I$ are not in $\mathscr{W}_{\{0\}}$. However, $\mathscr{W}_{\{0\}}$ has less obvious algebraic structures which can be described in the following way. Let $\mathcal{P} \subseteq \mathcal{B}(\mathscr{H})$ be a given set of operators. Then there exist the largest sets $\mathcal{Q}_{\mathcal{P}}, \mathcal{R}_{\mathcal{P}} \subseteq \mathcal{B}(\mathscr{H})$ such that $\mathcal{P}\mathcal{Q}_{\mathcal{P}} \subseteq \mathscr{W}_{\{0\}}$ and $\mathcal{R}_{\mathcal{P}}\mathcal{P} \subseteq \mathscr{W}_{\{0\}}$, where $\mathcal{P}\mathcal{Q}_{\mathcal{P}}$

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is the set of all products PQ with $P \in \mathcal{P}$ and $Q \in \mathcal{Q}_{\mathcal{P}}$; $\mathcal{R}_{\mathcal{P}}\mathcal{P}$ has a similar meaning. It follows from the part (i) of the following proposition that it is enough to study only one variant of the problem. We use the following notation: for $\mathcal{A} \subseteq \mathcal{B}(\mathscr{H})$ let $\mathcal{A}^* = \{A^*;\ A \in \mathcal{A}\}$.

Proposition 3. *Let $\mathcal{P}, \mathcal{P}_1$, and $\mathcal{P}_2$ be arbitrary non-empty subsets of $\mathcal{B}(\mathscr{H})$. Then*
(i) $(\mathcal{Q}_{\mathcal{P}})^* = \mathcal{R}_{\mathcal{P}^*}$;
(ii) if $I \in \mathcal{P}$, then $\mathcal{Q}_{\mathcal{P}} \subseteq \mathscr{W}_{\{0\}}$;
(iii) if $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $\mathcal{Q}_{\mathcal{P}_1} \supseteq \mathcal{Q}_{\mathcal{P}_2}$.

We omit a simple proof of this proposition.

For $\mathcal{P} = \mathcal{B}_+$, the set of all positive semidefinite operators on $\mathscr{H}$, we have an interesting description of $\mathcal{Q}_{\mathcal{B}_+}$. In the proof we need the following simple fact. If $F \subseteq \mathbb{C}$ is a nonempty set and $w \in \mathbb{C}$, then let $\operatorname{dist}(w, F) = \inf\{|w - z|;\ z \in F\}$ and, for $\varepsilon > 0$, let $F_\varepsilon = \{w \in \mathbb{C};\ \operatorname{dist}(w, F) \le \varepsilon\}$ denote the ε-hull of F. It is not hard to see that $\bigcap_{\varepsilon>0} F_\varepsilon = \overline{F}$.

Theorem 4. $\mathcal{Q}_{\mathcal{B}_+} = \{A \in \mathcal{B}(\mathscr{H});\ 0 \in \operatorname{conv}(\sigma(A))\}$.

Proof. Suppose that $0 \in \operatorname{conv}(\sigma(A))$ and let $P \in \mathcal{B}_+$. If A or P is not invertible, then $0 \in \sigma(PA) \subseteq \overline{W(PA)}$. Assume therefore that A and P are invertible. It follows that there exists $p > 0$ such that $\overline{W(P)} \subseteq [p, \infty)$. Since $0 \in \operatorname{conv}(\sigma(A))$ there exist $\lambda, \mu \in \partial\sigma(A)$ such that $0 = t\lambda + (1-t)\mu$ for some $t \in [0, 1]$. Numbers λ and μ are approximate eigenvalues of A, which means that there exist sequences $\{e_n\}_{n=1}^\infty$, $\{f_n\}_{n=1}^\infty \subseteq \mathcal{S}_{\mathscr{H}}$ such that $\lim_{n\to\infty} \|(A - \lambda I)e_n\| = 0$ and $\lim_{n\to\infty} \|(A - \mu I)f_n\| = 0$. Let m be a positive integer. Then there exists an index n_m such that $\|(A - \lambda I)e_n\| < \frac{1}{m}$ and $\|(A - \mu I)f_n\| < \frac{1}{m}$ for all $n \ge n_m$. Fix $n \ge n_m$ and denote $\omega_m = \langle Pe_n, e_n\rangle$, $\vartheta_m = \langle Pf_n, f_n\rangle$. Note that $\omega_m \ge p$ and $\vartheta_m \ge p$. One has

$$|\langle PAe_n, e_n\rangle - \lambda\omega_m| = |\langle P(A - \lambda I)e_n, e_n\rangle| \le \|P\|\|(A - \lambda I)e_n\| < \tfrac{\|P\|}{m}$$

and, similarly, $|\langle PAf_n, f_n\rangle - \mu\vartheta_m| < \|P\|/m$. Thus, $\lambda\omega_m$ and $\mu\vartheta_m$ are in the $\frac{\|P\|}{m}$-hull of $\overline{W(PA)}$. Since $\{\lambda\omega_m\}_{m=1}^\infty$ is a bounded sequence there exists a convergent subsequence, say $\{\lambda\omega_{m_k}\}_{k=1}^\infty$, which converges to $\lambda\omega$. It is obvious that this limit is in $\overline{W(PA)}$. Observe that $\omega \ge p$. The same reasoning gives $\vartheta \ge p$ such that $\mu\vartheta \in \overline{W(PA)}$. Denote $s_1 = t\vartheta/(t\vartheta + (1-t)\omega) \ge 0$ and $s_2 = (1-t)\omega/(t\vartheta + (1-t)\omega) \ge 0$. It is easily seen that $s_1 + s_2 = 1$ and $s_1(\lambda\omega) + s_2(\mu\vartheta) = 0$, which means that $0 \in \overline{W(PA)}$.

Assume now that $A \in \mathcal{Q}_{\mathcal{B}_+}$. Let $S \in \mathcal{B}(\mathscr{H})$ be an arbitrary invertible operator. Denote $P = S^*S \in \mathcal{B}_+$. Let $\varepsilon > 0$ be arbitrary. Since $0 \in \overline{W(PA)}$ there exists $x \in \mathcal{S}_{\mathscr{H}}$ (which may depend on ε) such that $|\langle PAx, x\rangle| < \varepsilon$. Let $y = \frac{1}{\|Sx\|}Sx \in \mathcal{S}_{\mathscr{H}}$. One has $|\langle SAS^{-1}y, y\rangle| = \|Sx\|^{-2}|\langle SAS^{-1}Sx, Sx\rangle| = \|Sx\|^{-2}|\langle PAx, x\rangle| < \|Sx\|^{-2}\varepsilon \le \|S^{-1}\|^2\varepsilon$. Since ε is arbitrary we conclude that $0 \in \overline{W(SAS^{-1})}$. As S is an arbitrary invertible operator we have, by Theorem 1, $0 \in \operatorname{conv}(\sigma(A))$. □

References

[1] J. Bračič, C. Diogo, *Operators with a given part of the numerical range*, Math. Slovaca, accepted.
[2] S. Hildebrandt, *Über den Numerischen Wertebereich eines Operators*, Math. Ann. **163** (1966), 230-247.

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