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ON A DYNAMIC CONTACT PROBLEM FOR A THERMOELASTIC PLATE

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Abstract. We deal with a dynamic contact problem for a thermoelastic plate vibrating against a rigid obstacle. Dynamics is described by a hyperbolic variational inequality for deflections. The plate is subjected to a perpendicular force and to a heat source. The parabolic equation for the change of the temperature contains the time derivative of the deflection. We formulate a weak solution of the system and verify its existence using the penalization method.

1. Introduction and notation

The dynamic contact problems are not frequently solved in the framework of variational inequalities. For the elastic problems there is only a very limited amount of results available (cf. [4] and there cited literature). We have solved these problems for geometrically nonlinear plates and shells in [2] and [3] respectively. We concentrate here on the linear Kirchhoff model of the plate subjected not only to the perpendicular forces but also to the temperature source. We shall use the model derived in [5] under the assumption of a small change of temperature compared with its reference temperature. In contrast to it the hyperbolic equation for the deflections is substituted here by the variational inequality. We formulate and solve the penalized initial-boundary value problem. Using the *a priori* estimates we achieve the sequence converging to a weak solution of the original problem.

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal or C^2 domain with a boundary Γ and $I \equiv (0,T)$ a bounded time interval, $Q = I \times \Omega,\ S = I \times \Gamma$. The unit outer normal vector is denoted by $\boldsymbol{n} = (n_1, n_2),\ \boldsymbol{\tau} = (-n_2, n_1)$ is the unit tangent vector. The constants $E > 0$ and $\nu \in [0, \frac{1}{2})$ are the Young modulus of elasticity and the Poisson ratio, respectively. We set

$$a = \frac{h^2}{12},\ b = \frac{Eh^2}{12\varrho(1-\nu^2)},$$

where h is the the plate thickness and ϱ is the density of the material.

With respect to a heat conduction we introduce following constants. The specific heat of the body $c > 0$, the coefficients of thermal conductivity $\lambda > 0$. Further we set $\alpha > 0$ the coefficient of thermal expanding and $\Upsilon > 0$ the reference temperature of the plate. We shall use the abbreviations

$$\kappa = \frac{\lambda}{\rho c} > 0,\ d = \frac{\kappa 12}{h^2} > 0,\ e = \frac{\kappa\alpha^2\Upsilon E}{\lambda(1-2\nu)} > 0.$$

We shall employ the following notations for space and time derivatives are

$$\frac{\partial}{\partial s} \equiv \partial_s,\ \frac{\partial^2}{\partial s \partial r} \equiv \partial_{sr},\ \partial_i = \partial_{x_i},\ i = 1,2;\ \dot{v} = \frac{\partial v}{\partial t},\ \ddot{v} = \frac{\partial^2 v}{\partial t^2},\ v : Q \mapsto \mathbb{R}.$$

For a domain or an appropriate manifold M and $p \geq 1$ we define the Banach space $L_p(M)$ of real valued measurable functions with integrable power of p. The space $L_\infty(M)$ is the Banach space of essentially bounded functions. By $H^k(M) \subset L_2(M)$ with $k \geq 0$ we denote the Sobolev (for

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a noninteger k the Sobolev-Slobodetskii) spaces of functions defined on M. For the anisotropic spaces $H^k(M)$, $k=(k_1,k_2)\in\mathbb{R}^2_+$, k_1 is related with the time while k_2 with the space variables provided M is a time-space domain.

By $\mathring{H}^1(\Omega)$ we denote the subspace of functions from $H^1(\Omega)$ with zero traces on Γ. The dual to $\mathring{H}^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ with $\langle\cdot,\cdot\rangle$ the duality pairing between $H^{-1}(\Omega)$ and $\mathring{H}^1(\Omega)$.

2. Formulation of the problem

A triple $\{u,g,\theta\}$ expresses an unknown deflection of the middle plane, an unknown contact force between the plate and the rigid obstacle and an unknown change of the temperature. Classical formulation for the plate simply supported, with the zero change of the temperature on the boundary and acting under a perpendicular load f and the heat source p is composed of the system

$$\left.\begin{aligned}&\ddot u - a\Delta\ddot u + b(\triangle^2 u + \tfrac{1+\nu}{2}\triangle\theta) = f+g,\\ &u\ge 0,\ g\ge 0,\ ug=0,\\ &\dot\theta-\kappa\triangle\theta+d\theta-e\triangle\dot u = p\end{aligned}\right\}\ \text{on } Q, \tag{1}$$

the boundary conditions

$$u=w,\ \theta=M(u,\theta)=0,\ u\ge 0,\ g\ge 0,\ ug=0\ \text{ on } S, \tag{2}$$

$$M(u,\theta)\equiv b\big(\triangle u+(1-\nu)(2n_1n_2\partial_{12}u-n_1^2\partial_{22}u-n_2^2\partial_{11}u)+\tfrac{1+\nu}{2}\theta\big)$$

and the initial conditions

$$u(0,\cdot)=u_0,\ \dot u(0,\cdot)=v_0,\ \theta(0,\cdot)=\theta_0 \text{ on } \Omega. \tag{3}$$

For $u,y\in L_2(I;H^2(\Omega))$ we define the following bilinear form

$$A:(u,y)\mapsto b\big(\partial_{11}u\partial_{11}y+\partial_{22}u\partial_{22}y+\nu(\partial_{11}u\partial_{22}y+\partial_{22}u\partial_{11}y)+2(1-\nu)\partial_{12}u\partial_{12}y\big) \tag{4}$$

almost everywhere on Q and introduce for a fixed function $w:\Omega\mapsto\mathbb{R}$ a shifted cone

$$\mathscr{K}:=\{y\in w+L_\infty(I;V);\ \dot y\in L_\infty(I;\mathring{H}^1(\Omega)),\ y\ge 0 \text{ on } Q\}, \tag{5}$$

where

$$V=H^2(\Omega)\cap\mathring{H}^1(\Omega). \tag{6}$$

Then the variational formulation of (1–3) has the following form:

Problem $\mathscr{P}$. *Look for* $\{u,\theta\}\in\mathscr{K}\times(L_\infty(I;L_2(\Omega))\cap L_2(I;\mathring{H}^1(\Omega)))$ *such that* $\dot u(T,\cdot)\in\mathring{H}^1(\Omega)$, $\dot\theta\in L_2(I;H^{-1}(\Omega))$, *the relations*

$$\begin{aligned}&\int_Q\big(A(u,y-u)-\dot u(\dot y-\dot u)-a\nabla\dot u\cdot\nabla(\dot y-\dot u)-b\tfrac{1+\nu}{2}\nabla\theta\cdot\nabla(y-u)\big)\,dx\,dt\\ &+\int_\Omega(\dot u(y-u)+\nabla\dot u\cdot\nabla(y-u))\,(T,\cdot)\,dx\\ &\ge\int_\Omega(v_0(y(0,\cdot)-u_0)+\nabla v_0\cdot\nabla(y(0,\cdot)-v_0))\,dx+\int_Q f(y-u)\,dx\,dt,\end{aligned} \tag{7}$$

$$\int_I\langle\dot\theta,z\rangle\,dt+\int_Q(d\theta z+\kappa\nabla\theta\cdot\nabla z+e\nabla\dot u\cdot\nabla z)\,dx\,dt=\int_Q pz\,dx\,dt \tag{8}$$

hold for any $\{y,z\}\in\mathscr{K}\times L_2(I;\mathring{H}^1(\Omega))$ *and the initial conditions* (3) *are fulfilled (for* $\dot u$ *in certain generalized sense).*

Problem $\mathscr{P}$ will be solved under following assumptions

$$
\begin{aligned}
&w \in H^2(\Omega),\ w \geq w_0 > 0 \text{ on } \Omega;\ w_{|\Gamma} = u_{0|\Gamma},\\
&u_0 \in H^2(\Omega),\ u_0 \geq 0 \text{ on } \Omega;\ v_0 \in \mathring{H}^1(\Omega),\ \theta_0 \in L_2(\Omega),\ \{f,p\} \in L_2(Q)^2.
\end{aligned} \tag{9}
$$

3. Penalized problem

For any $\eta > 0$ we formulate the *penalized problem*

$$
\left.
\begin{aligned}
&\ddot{u} - a\triangle\ddot{u} + b(\triangle^2 u + \tfrac{1+\nu}{2}\triangle\theta) = f + \eta^{-1}u^-,\\
&\dot{\theta} - \kappa\triangle\theta + d\theta - e\triangle\dot{u} = p
\end{aligned}
\right\} \text{ on } Q, \tag{10}
$$

$$
u = w,\ \theta = M(u,\theta) = 0 \text{ on } S \tag{11}
$$

and the initial conditions (3) *hold.*

It has the following variational formulation.

Problem $\mathscr{P}_\eta$. *Look for* $\{u,\theta\} \in (w+L_\infty(I;V))\times L_\infty(I;\mathring{H}^1(\Omega))$ *such that* $\{\dot{u},\dot{\theta}\} \in L_\infty(I;\mathring{H}^1(\Omega))\times L_2(I;(H^{-1}(\Omega)),\ \ddot{u} \in L_2(Q)$, *the equations*

$$
\int_Q \left(\ddot{u}(y - a\triangle y) + A(u,y) - b\tfrac{1+\nu}{2}\nabla\theta\cdot\nabla y - \eta^{-1}u^- y\right)\,dx\,dt = \int_Q fy\,dx\,dt, \tag{12}
$$

$$
\int_I \langle\dot{\theta}, z\rangle\,dt + \int_Q (d\theta z + \kappa\nabla\theta\cdot\nabla z + e\nabla\dot{u}\cdot\nabla z)\,dx\,dt = \int_Q pz\,dx\,dt \tag{13}
$$

hold for any $\{y,z\} \in L_2(I;V)\times L_2(I;\mathring{H}^1(\Omega))$ *and the initial conditions* (3) *remain.*

We shall verify the existence of a solution to the penalized problem.

Theorem 3.1. *For every* $\eta > 0$ *there exists a solution* $\{u,\theta\}$ *of the problem* $\mathscr{P}_\eta$.

Proof. Let us denote by $\{v_i \in V;\ i \in \mathbb{N}\}$ a basis of V orthonormal with respect to the inner product

$$
(u,v)_a = \int_\Omega (uv + a\nabla u\cdot\nabla v)\,dx,\ u,\ v \in \mathring{H}^1(\Omega)
$$

and by $\{w_i \in \mathring{H}^1(\Omega);\ i \in \mathbb{N}\}$ an orthonormal in $L_2(\Omega)$ basis of $\mathring{H}^1(\Omega)$.

We construct the Galerkin approximation $\{u_m,\theta_m\}$ of a solution in a form

$$
u_m(t) = w + \sum_{j=1}^m \alpha_j(t)v_j,\ \theta_m(t) = \sum_{j=1}^m \beta_j(t)w_j;\ \{\alpha_j(t),\beta_j(t)\} \in \mathbb{R}^2,\ j = 1,...,m,
$$

$$
\int_\Omega \left(\ddot{u_m}v_i + a\,\nabla\ddot{u_m}\cdot\nabla v_i + A(u_m,v_i) - b\tfrac{1+\nu}{2}\nabla\theta_m\cdot\nabla v_i - \eta^{-1}u_m^- v_i\right)\,dx = \int_\Omega fv_i\,dx, \tag{14}
$$

$$
\int_\Omega \left(\dot{\theta}_m w_i + \kappa\nabla\theta_m\cdot\nabla w_i + d\theta_m w_i + e\nabla\dot{u}_m\cdot\nabla w_i\right)\,dx = \int_\Omega pw_i\,dx,\ i = 1,...,m, \tag{15}
$$

$$
\begin{aligned}
&u_m(0) = u_{0m},\ u_{0m} \to u_0 \text{ in } H^2(\Omega); \dot{u}_m(0) = v_{0m},\ v_{0m} \to v_0 \text{ in } \mathring{H}^1(\Omega);\\
&\theta_m(0) = \theta_{0m},\ \theta_{0m} \to \theta_0 \text{ in } L_2(\Omega).
\end{aligned} \tag{16}
$$

The initial value problem (14)-(16) fulfils the conditions for the local existence of solution $\{u_m,\theta_m\}$ on some interval $I_m \equiv [0,t_m],\ 0 < t_m < T$.

Let us set $\gamma = b\frac{1+\nu}{2e}$. To derive the *a priori* estimates for solutions of (14)-(16) we multiply the equations (14) by $\dot\alpha_i(t)$ and (15) by $\gamma\beta_i(t)$ respectively, add with respect to i and integrate on $[0, t_m]$. We obtain for $Q_m := I_m \times \Omega$

$$\int_{Q_m} \left[\frac{1}{2}\partial_t \left(\dot u_m^2 + a|\nabla \dot u_m|^2 + A(u_m, u_m) + \gamma\theta_m^2 + \eta^{-1}(u_m^-)^2\right) + \gamma(\kappa|\nabla\theta_m|^2 + d\theta_m^2)\right] dx\, dt$$
$$= \int_{Q_m} (f\dot u_m + \gamma\, p\, \theta_m)\, dx\, dt$$

which leads to the estimate

$$\begin{aligned} &\|\dot u_m\|^2_{L_\infty(I;\mathring{H}^1(\Omega))} + \|u_m\|^2_{L_\infty(I;V)} + \|\theta_m\|^2_{L_\infty(I;L_2(\Omega))} + \|\theta_m\|^2_{L_2(I;\mathring{H}^1(\Omega))} \\ &+ \eta^{-1}\|u_m^-\|^2_{L_\infty(I;L_2(\Omega))} \le C_1 \equiv C_1(f, p, u_0, v_0, \theta_0). \end{aligned} \tag{17}$$

The prolongation to the whole interval I is due to the original estimate for I_m not depending on m.

From the equation (15) we obtain straightforwardly the estimate

$$\|\dot\theta_m\|_{L_2(I;W_m{}^*)} \le C_2(f, p, u_0, v_0, \theta_0),\ m \in \mathbb{N}, \tag{18}$$

where $W_m \subset \mathring{H}^1(\Omega)$ is the linear hull of $\{w_i\}_{i=1}^m$.

From (14) we obtain

$$\|\ddot u_m - a\triangle \ddot u_m\|^2_{L_2(I;V_m{}^*)} \le C_3(\eta),\ m \in \mathbb{N}, \tag{19}$$

where $V_m \subset H^2(\Omega)$ is the linear hull of $\{v_i\}_{i=1}^m$.

We proceed with the convergence of the Galerkin approximation. Applying the estimate (17), the compact imbedding theorem and interpolation in Sobolev spaces we obtain subsequences of $\{u_m\}$, $\{\theta_m\}$ (denoted again by $\{u_m\}$, $\{\theta_m\}$), and functions u, θ with the convergences

$$\begin{aligned} u_m &\rightharpoonup^* u && \text{in } L_\infty(I; V), \\ \dot u_m &\rightharpoonup^* \dot u && \text{in } L_\infty(I; \mathring{H}^1(\Omega)), \\ u_m &\to u && \text{in } C(I; H^{1-\varepsilon}(\Omega)) \cap L_\infty(I; H^{2-\varepsilon}(\Omega)) \text{ for any } \varepsilon > 0,, \\ \theta_m &\rightharpoonup^* \theta && \text{in } L_\infty(I; L_2(\Omega)) \cap L_2(I; \mathring{H}^1(\Omega)). \end{aligned} \tag{20}$$

The estimates (18), (19) imply the convergence

$$\dot\theta_m \rightharpoonup \dot\theta \quad \text{in } L_2(I; W^*), \tag{21}$$

$$(\ddot u_m - a\triangle \ddot u_m) \rightharpoonup (\ddot u - a\triangle \ddot u) \quad \text{in } L_2(I; Y^*), \tag{22}$$

where $W = \bigcup_{m\in\mathbb{N}} W_m$, $\overline{W} = \mathring{H}^1(\Omega)$ and $Y = \bigcup_{m\in\mathbb{N}} V_m$, $\overline{Y} = V$.

The convergences (21), (22) imply

$$\|\dot\theta_m\|_{L_2(I;H^{-1}(\Omega))} \le C_2(f, p, u_0, v_0, \theta_0),\ m \in \mathbb{N}, \tag{23}$$

$$\dot\theta_m \rightharpoonup \dot\theta \quad \text{in } L_2(I; H^{-1}(\Omega)), \tag{24}$$

$$\|\ddot u_m - a\triangle \ddot u_m\|^2_{L_2(I;Y^*)} \le C_3(\eta),\ m \in \mathbb{N}. \tag{25}$$

Moreover we obtain from (25) a better acceleration estimate

$$\|\ddot u_m\|_{L_2(Q)} \le C_4(\eta) \tag{26}$$

and the convergence

$$\ddot u_m \rightharpoonup \ddot u \text{ in } L_2(Q) \tag{27}$$

for a chosen subsequence denoted again by $\{\ddot u_m\}$. We have applied also the properties of the elliptic operator $v \mapsto v - a\triangle v,\ v \in V$; in the same way as in [1] setting

$$\|\ddot u_m\|_{L_2(Q)} = \sup_{\|f\|_{L_2(Q)}\le 1} \left|\int_Q \ddot u_m\, f\, dx\, dt\right| \le c \sup_{\|v\|_{L_2(I;V)}\le 1} \left|\int_Q \ddot u_m (v - a\triangle v)\, dx\, dt\right| \le C_4(\eta).$$

Let $\mu \in \mathbb{N},\ y_\mu = \sum_{i=1}^{\mu} \phi_i(t)v_i,\ z_\mu = \sum_{i=1}^{\mu} \phi_i(t)w_i,\ \phi_i \in \mathscr{D}(0,T),\ i = 1,...,\mu$. We have for arbitrary $t \in I$ the relations

$$\int_\Omega \left(\ddot{u_m}(y_\mu - a\triangle y_\mu) + A(u_m, y_\mu) - b\tfrac{1+\nu}{2}\nabla\theta_m \cdot \nabla y_\mu - \eta^{-1}u_m^- y_\mu)\right) dx = \int_\Omega f y_\mu \, dx,$$
$$\int_\Omega \left(\dot{\theta}_m z_\mu + \kappa\nabla\theta_m \cdot \nabla z_\mu + d\theta_m z_\mu + e\nabla\dot{u}_m \cdot \nabla z_\mu\right) dx = \int_\Omega p z_\mu \, dx,\ \forall\ m \geq \mu,\ t \in I.$$

The convergences (20), (24), (27) imply that functions u, θ fulfil

$$\int_\Omega \left(\ddot{u}(y_\mu - a\triangle y_\mu) + A(u, y_\mu) - b\tfrac{1+\nu}{2}\nabla\theta \cdot \nabla y_\mu - \eta^{-1}u^- y_\mu)\right) dx = \int_\Omega f y_\mu \, dx, \tag{28}$$
$$\int_\Omega \left(\dot{\theta} z_\mu + \kappa\nabla\theta \cdot \nabla z_\mu + d\theta z_\mu + e\nabla\dot{u} \cdot \nabla z_\mu\right) dx = \int_\Omega p z_\mu \, dx. \tag{29}$$

Functions $\{y_\mu\},\ \{z_\mu\}$ form a dense subsets of the spaces $L_2(I;V)$ and $L_2(I;\mathring{H}^1(\Omega))$ respectively. Then we obtain from (28), (29) the relations (12), (13). The initial conditions (3) follow due to (16) and the proof of the existence of a solution is complete.

4. Solvability of the original problem

The estimates (17), (23) imply the following η independent estimates :

$$\begin{aligned}&\|\dot{u}_\eta\|^2_{L_\infty(I;\mathring{H}^1(\Omega))} + \|u_\eta\|^2_{L_\infty(I;V)} + \|\theta_\eta\|^2_{L_\infty(I;L_2(\Omega))} + \|\theta_\eta\|^2_{L_2(I;\mathring{H}^1(\Omega))} + \|\dot{\theta}_\eta\|^2_{L_2(I;H^{-1}(\Omega))}\\ &+ \eta^{-1}\|u_\eta^-\|^2_{L_\infty(I;L_2(\Omega))} \leq c \equiv c(f,p,u_0,u_1,\theta_0).\end{aligned} \tag{30}$$

for a solution $\{u_\eta, \theta_\eta\},\ \eta > 0$ of the penalized problem.

The acceleration term $\ddot{u}_\eta$ does not appear in (30). It is then suitable to transform the penalized relation (12) using the method by parts with respect to t and the Gauss formula with respect to x. We obtain the system

$$\begin{aligned}&\int_Q \left(A(u_\eta, y) - \dot{u_\eta}\dot{y} - a\nabla\dot{u_\eta} \cdot \nabla\dot{y} - b\tfrac{1+\nu}{2}\nabla\theta_\eta \cdot \nabla y\right) dx\, dt + \int_\Omega \left(\dot{u_\eta} y + a\nabla\dot{u_\eta} \cdot \nabla y\right)(T,\cdot)\, dx\\ &= \int_\Omega \left(v_0 y(0,\cdot) + a\nabla v_0 \cdot \nabla y(0,\cdot)\right) dx + \int_Q (f + \eta^{-1}u_\eta^-) y \, dx\, dt,\end{aligned} \tag{31}$$

$$\int_Q \left(\dot{\theta}_\eta z + \kappa\nabla\theta_\eta \cdot \nabla z + d\theta_\eta z + e\nabla\dot{u}_\eta \cdot \nabla z\right) dx\, dt = \int_Q p y \, dx\, dt \tag{32}$$

holding for any $\{y, z\} \in L_2(I;V) \times L_2(I;\mathring{H}^1(\Omega))$ with $\dot{y} \in L_2(I;\mathring{H}^1(\Omega))$.

We derive an $\eta-$independent estimate of the penalty term $\eta^{-1}u_\eta^-$. Applying the assumptions (9) and the definition of u_η^- we obtain

$$0 \leq w_0 \int_Q \eta^{-1}u_\eta^- \, dx\, dt \leq \int_Q \eta^{-1}u_\eta^- w \, dx\, dt \leq \int_Q \eta^{-1}u_\eta^- (w - u_\eta) \, dx\, dt.$$

After inserting $y = w - u_\eta$ in (31) we achieve using the estimates (30) the crucial estimate

$$\|\eta^{-1}u_\eta^-\|_{L_1(Q)} \leq C \equiv C(f,p,u_0,u_1,\theta_0). \tag{33}$$

Hence there exists a sequence $\eta_k \searrow 0$, functions $\{u, \theta\}$ and a functional g such that for $\{u_k, \eta_k\} \equiv \{u_{\eta_k}, \theta_{\eta_k}\}$ the following convergences hold:

$$
\begin{aligned}
u_k &\rightharpoonup^* u && \text{in } L_\infty(I;V),\\
\dot u_k &\rightharpoonup^* \dot u && \text{in } L_\infty(I;\mathring{H}^1(\Omega)),\\
\dot u_k(T,\cdot) &\rightharpoonup \dot u(T,\cdot) && \text{in } \mathring{H}^1(\Omega),\\
u_k &\to u && \text{in } C(I;H^{1-\varepsilon}(\Omega)) \cap L_\infty(I;H^{2-\varepsilon}(\Omega)) \text{ for any } \varepsilon > 0,\\
\eta^{-1}u_k^- &\rightharpoonup^* g && \text{in } (L_\infty(Q))^*,\\
\theta_k &\rightharpoonup^* \theta && \text{in } L_\infty(I;L_2(\Omega)) \cap L_2(I;\mathring{H}^1(\Omega)),\\
\dot\theta_k &\rightharpoonup \dot\theta && \text{in } L_2(I;H^{-1}(\Omega))
\end{aligned}
\tag{34}
$$

The convergences above prove the relation (8). Together with (31) they imply

$$
\int_Q \left(A(u,y) - \dot u\dot y - a\nabla\dot u\cdot\nabla\dot y - b\tfrac{1+\nu}{2}\nabla\theta\cdot\nabla y\right)dx\,dt + \int_\Omega (\dot u y + a\nabla\dot u\cdot\nabla y)(T,\cdot)\,dx
$$
$$
= \int_\Omega (v_0 y(0,\cdot) + a\nabla v_0\cdot\nabla y(0,\cdot))\,dx + \int_Q fy\,dx\,dt + \langle\langle g, y\rangle\rangle
$$

for any $y \in L_2(I;V)$ with $\dot y \in L_2(I;\mathring{H}^1(\Omega))$, where $\langle\langle\cdot,\cdot\rangle\rangle$ is the duality pairing between $(L_\infty(Q))^*$ and $L_\infty(Q)$.

We have the orthogonality

$$
\langle\langle g, u\rangle\rangle = 0
$$

due to the relations $\langle\langle g,u\rangle\rangle = \lim_{k\to\infty}\eta_k^{-1}\|u_k^-\|^2_{L_2(Q)} = 0$.

The relations $\langle\langle g,y\rangle\rangle = \lim_{k\to\infty}\int_Q \eta_k^{-1}u_k^- y\,dx\,dt \ge 0$ for any $y \in \mathscr{K}$ imply together with the orthogonality proved above that the variational inequality (7) is fulfilled and we have verified the existence theorem:

Theorem 4.1. *Let the assumptions* (9) *hold. Then there exists a solution of Problem* $\mathscr{P}$.

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