WHICH POSITIVE OPERATOR CAN BE THE ASYMPTOTIC LIMIT OF A CONTRACTION OR A POWER-BOUNDED OPERATOR?

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Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ stand for the C*-algebra of bounded, linear operators on $\mathcal{H}$. The operator $T \in \mathcal{B}(\mathcal{H})$ is a contraction if $\|T\| \leq 1$. Let us consider the sequence $\{T^{*n}T^n\}_{n=1}^{\infty}$ of positive operators, which is decreasing, so it has a limit in the strong operator topology (SOT):

$$A_T := \lim_{n\to\infty} T^{*n}T^n.$$

We say that A_T *arises asymptotically from* T, or A_T is the *asymptotic limit* of T. In the case when this convergence holds in norm, we say that A_T *arises asymptotically from* T *in uniform convergence* or A_T is the *uniform asymptotic limit* of T. We recall that $A_T^{1/2}$ acts as an intertwining mapping in a canonical realization of the so called unitary and isometric asymptote of the contraction T, which is a very efficient tool in the theory of Hilbert space contractions.

A generalization of the above setting is as follows: let us fix a Banach limit L and consider a power-bounded operator $T \in \mathrm{PWB}(\mathcal{H})$ (i. e. $\sup\{\|T^n\| \colon n \in \mathbb{N}\} < \infty$). Easy to see that the following is a bounded sesqui-linear functional:

$$w_{T,L}(h,k) := L - \lim_{n\to\infty}\langle T^n h, T^n k\rangle,$$

hence there exists a unique positive operator $A_{T,L}$ which represents it, i. e. that $w_{T,L}(h,k) = \langle A_{T,L}h, k\rangle$ is satisfied for all $h, k \in \mathcal{H}$. We call this positive operator the *L-asymptotic limit* of the power-bounded operator T. If $T^{*n}T^n$ converges in SOT, then usually we just write A_T instead of $A_{T,L}$ and we say that A_T *arises from the power-bounded operator* T or that the power-bounded operator T has the *asymptotic limit* A_T.

It can happen that the Cesaro means of the sequence converge in SOT to a positive operator $A_{T,C}$, in that case we call $A_{T,C}$ the *Cesaro asymptotic limit* of T. We note that usually $A_{T,C} \neq A_{T,L}$.

The main aim of the talk is to discuss some classes of the following type:

$$\{A_T \colon T \in \mathcal{B}(\mathcal{H}), \|T\| \leq 1\},\ \{A_T \colon T \in \mathrm{PWB}(\mathcal{H})\},$$
$$\{A_T \colon T \in \mathcal{B}(\mathcal{H}), \|T\| \leq 1, \|T^{*n}T^n - A_T\| \to 0\},\ \{A_T \colon T \in \mathrm{PWB}(\mathcal{H})\},$$
$$\{A_{T,c} \colon T \in \mathrm{PWB}(\mathcal{H})\},\ \{A_{T,L} \colon L \text{ is a Banach limit and } T \in \mathrm{PWB}(\mathcal{H})\},$$
$$\{A_T \colon T \in \mathrm{PWB}(\mathcal{H}) \text{ is similar to a normal operator}\}\ldots$$

and so on. The finite dimensional case is quite interesting, so it will be examined as well.

We also investigate some connections between operators which have the same asymptotic limit.

References

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