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ORTHOGONAL PROJECTION IN A UNIFORMLY CONVEX BANACH SPACES

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Abstract. In a uniformly convex Banach space with a continuous generalized semi-inner product X we investigate the relation of orthogonality in X and projections acting on X. We prove that the decomposition theorem holds on a uniformly convex Banach space with a continuous generalized semi-inner product . This result is presented in detail in Theorem 4. The main results in this paper is Theorem 8. We prove the uniqueness of orthogonal linear projections. For more results we refer to [1], [2], [3], [4] and references therein.

1. Generalized Semi-inner Products

Let X be a vector space over $\mathbb{C}$. We call a function $[\cdot,\cdot]_\varphi : X\times X \to \mathbb{C}$ *a generalized semi-inner product* (g.s.i.p.) on a vector space X if it satisfies the following conditions:
(i) Linearity with respect to the first variable:

$$[\alpha x+\beta y, z]_\varphi = \alpha[x,z]_\varphi + \beta[y,z]_\varphi \quad \text{for all} \quad \alpha,\beta\in\mathbb{C} \quad \text{and} \quad x,y,z\in X;$$

(ii) Positivity: $[x,x]_\varphi > 0$ for all $x\in X\setminus\{0\}$;
(iii) A generalization of the Cauchy-Schwarz inequality: there holds for some $\varphi,\ \psi : \mathbb{R}_+ \to \mathbb{R}_+$ that

$$|[x,y]_\varphi| \le \varphi([x,x]_\varphi)\psi([y,y]_\varphi), \quad x,y\in X$$

and the equality holds when $x=y$.

We require that

$$\varphi(0)=0, \quad \varphi(t)>0 \quad \text{for} \quad t>0$$

and

$$\psi(0)=0, \quad \psi(t)=\frac{t}{\varphi(t)} \quad \text{for} \quad t>0.$$

The importance of a generalized semi-inner product space (s.i.p.s.) is that every normed vector space can be represented as a semi-inner product space so that the theory of operators on a Banach space can be penetrated by Hilbert space type arguments.

Theorem 1. [5] *Let $[\cdot,\cdot]_\varphi$ be a g.s.i.p. on a vector space X over $\mathbb{C}$. Then $\|x\|_\varphi = \varphi([x,x]_\varphi)$ defines a norm on X.*

Moreover let X be a vector space equipped with the norm $\|\cdot\|$. If φ is surjective onto $\mathbb{R}_+$, then there exists a g.s.i.p. on X such that $\|\cdot\|_\varphi = \|\cdot\|$.

In a normed vector space X we set $S=\{x\in X:\ \|x\|=1\}$. We introduce additional properties of a g.s.i.p.

A very convenient property of a g.s.i.p. is continuity with respect to the second variable.

A g.s.i.p. space X is called a *continuous g.s.i.p. space* when a generalized semi-inner product satisfies the following additional condition:
For every $x,y\in S$,

$$\mathrm{Re}[y,x+\lambda y]_\varphi \to \mathrm{Re}[y,x]_\varphi \quad \text{for all real} \quad \lambda\to 0. \tag{1}$$

Define a relation on a s.i.p. space which may be called an orthogonality relation. Let $x,y\in X$. We say that x is *orthogonal* to y and y is *transversal* to x if $[y,x]_\varphi = 0$.

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On a normed space we also can study the orthogonality relation (in the sense of Birkhoff) defined as follows:

A vector x is *orthogonal* to y in the sense of Birkhoff if

$$\|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

One more piece of notation: throughout the paper we write $\|x\| = \varphi([x, x]_\varphi)$ and $\|x\|_* = \psi([x, x]_\varphi)$.

It is worth noting that orthogonality in the sense of Birkhoff is very close to the concept of an element of best approximation. In a continuous g.s.i.p. space orthogonality relation is equivalent to Birkhoff orthogonality relation.

Lemma 2. *In a continuous g.s.i.p.s. x is orthogonal to y if and only if x is orthogonal to y in the sense of Birkhoff.*

Proof. Let x be normal to y. Thus,

$$\|x + \lambda y\|\|x\|_* \geq |[x + \lambda y, x]_\varphi| = \\ |[x, x]_\varphi + \lambda[y, x]_\varphi| = \|x\|\|x\|_*.$$

Therefore, $||x + \lambda y|| \geq ||x||$ for all complex λ.

Let $[\cdot, \cdot]_\varphi$ be a continuous g.s.i.p. If $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$, then

$$0 \leq \|x + \lambda y\|_* \|x + \lambda y\| - \|x + \lambda y\|_* \|x\| \leq \\ [x + \lambda y, x + \lambda y]_\varphi - |[x, x + \lambda y]_\varphi| \leq \\ \mathrm{Re}[x, x + \lambda y]_\varphi + \mathrm{Re}\,\{\lambda[y, x + \lambda y]_\varphi\} - \mathrm{Re}[x, x + \lambda y]_\varphi.$$

Therefore, $\mathrm{Re}\,\{\lambda[y, x + \lambda y]_\varphi\} \geq 0$.
For real λ we have

$$\mathrm{Re}[y, x + \lambda y]_\varphi \geq 0 \quad \text{for} \quad \lambda \geq 0;$$
$$\mathrm{Re}[y, x + \lambda y]_\varphi \leq 0 \quad \text{for} \quad \lambda \leq 0.$$

From the continuity condition, for real λ, we have

$$\lim_{\lambda \to 0} \mathrm{Re}[y, x + \lambda y]_\varphi = \mathrm{Re}[y, x]_\varphi$$

through positive values for $\lambda \to 0^+$ and through negative values for $\lambda \to 0^-$. Thus $\mathrm{Re}[y, x]_\varphi = 0$. For imaginary λ, say $\lambda = i\lambda_1$ with λ_1 real, we obtain $\mathrm{Re}[iy, x]_\varphi = 0$, i.e. $\mathrm{Im}[y, x]_\varphi = 0$. Therefore, $[y, x]_\varphi = 0$. ☐

For the purpose of studing projections, we characterize strict convexity of X in terms of the g.s.i.p.

Theorem 3. [5] *The normed vector space X is striclty convex if and only if whenever*

$$[x, y]_\varphi = \varphi([x, x]_\varphi)\frac{[y, y]_\varphi}{\varphi([y, y]_\varphi)}, \quad x, y \neq 0$$

then $y = \alpha x$ for some $\alpha > 0$.

To extend Hilbert space type argument we prove the decomposition theorem. The desired statement follows after one impose an additional structure on a g.s.i.p. chiefly to guarantee the existence of orthogonal vectors to closed subspaces.

Theorem 4. *Let X be a uniformly convex Banach space with a continuous generalized semi-inner product. Let M be a closed subspace of X. Then each $x \in X$ can be uniquely decomposed in the form $x = y + z$ with $y \in M$ and $z \in M^\perp = \{u \in X \,:\, \forall\, v \in M \;\; [v, u]_\varphi = 0\,\}$.*

Proof. It is well known that, in a uniformly convex Banach space, for a closed vector subspace M and a vector $x \notin M$, there exists a unique nonzero vector $y \in M$ such that

$$||x - y|| = d(x, M) = \inf\{||x - y'|| : \, y' \in M\}.$$

Let us set $z = x - y$. Then z is orthogonal to M.

In order to prove the uniqueness of the representation $x = y+z$ we assume that $x = y_1+z_1 = y_2 + z_2$ where $y_1, y_2 \in M$ and $z_1, z_2 \in M^{\perp}$. It follows that $z_1 - z_2 = y_1 - y_2 \in M$. If $z_1 - z_2 \in M \cap M^{\perp}$, then $z_1 - z_2 = 0$ and $y_1 = y_2$. If $z_1 - z_2 \notin M^{\perp}$, then

$$0 = [z_1 - z_2, z_1]_{\varphi} = [z_1, z_1]_{\varphi} - [z_2, z_1]_{\varphi} \geq \|z_1\|\|z_1\|_* - \|z_2\|\,\|z_1\|_*,$$
$$0 = [z_2 - z_1, z_2]_{\varphi} = [z_2, z_2]_{\varphi} - [z_1, z_2]_{\varphi} \geq \|z_2\|\|z_2\|_* - \|z_1\|\,\|z_2\|_*.$$

Therefore,

$$\|z_1\| = \|z_2\| \text{ and } \|z_1\|\,\|z_2\|_* = [z_1, z_2]_{\varphi}.$$

By the strict convexity of X, we obtain $z_1 = z_2$. This implies that $y_1 = y_2$. □

2. Orthogonal Projections

Let X be a uniformly convex Banach space with a continuous generalized semi-inner product and let M be a closed subspace of X. Let $P\colon X \to M$ be a linear projection. We say that P is *orthogonal* if $(\ker P)^{\perp} = M$.

Then the following theorem holds.

Theorem 5. *Let M be a closed subspace of a uniformly convex Banach space X with a continuous generalized semi-inner product. Let $P : X \to M$ be a linear projection. If $\|P\| = 1$, then P is orthogonal.*

Proof. We shall show that $(\ker P)^{\perp} = M$. For $x \in X$ we have

$$\|P(x)\| \leq \|x\|. \tag{2}$$

Setting x equal to $Px + \lambda(y - Py)$ in (2) we obtain

$$\|P(Px + \lambda(y - Py))\| \leq \|Px + \lambda(y - Py)\|,$$

hence

$$\|Px\| \leq \|Px + \lambda(y - Py)\|.$$

By virtue of Theorem 2, it is equivalent to the fact that Px is orthogonal to every $z \in \ker P$.

Conversely, suppose that $x \in (\ker P)^{\perp}$. Then $[z, x]_{\varphi} = 0$ for $z \in \ker P$. Hence $[x - Px, x]_{\varphi} = 0$ and

$$\|x\|\|x\|_* = [x - Px + Px, x]_{\varphi} = [x - Px, x]_{\varphi} + [Px, x]_{\varphi} \leq \|Px\|\|x\|_* \leq \|x\|\|x\|_*.$$

By assumptions it follows that $\|x\| = \|Px\|$ and $\|Px\|\|x\|_* = [Px, x]$. By the strict convexity of X, we obtain $Px = x$, and so $x \in M$. □

Now we can conclude that in a uniformly convex Banach space with a continuous g.s.i.p. every orthogonal linear projection is a projection of norm one.

Theorem 6. *Assume that X is a uniformly convex Banach space with a continuous generalized semi-inner product and M is a closed subspace of X. Let $P\colon X \to M$ be a linear projection. If P is orthogonal, then the norm of P is equal to one.*

Proof. Let $x \in X$. Then $Px - x \in \ker P$ and

$$\begin{aligned}\|Px\|\|Px\|_* = [Px, Px]_{\varphi} &= [Px - x + x, Px]_{\varphi} = \\ &= [Px - x, Px]_{\varphi} + [x, Px]_{\varphi} = [x, Px]_{\varphi}.\end{aligned}$$

Using the generalization of the Cauchy-Schwarz inequality we get

$$\|Px\| \leq \|x\|,$$

hence $\|P\| = 1$. □

Lewicki and Skrzypek proved that minimal projections onto symmetric subspaces of smooth Banach spaces are unique (see [2]). Now, we show an analogous theorem in a uniformly convex Banach space X with a continuous g.s.i.p. In its proof we use the structure of a generalized semi-inner product.

Lemma 7. *Let $P\colon X \to M$ be an ortogonal projection. Then P is a unique orthogonal projection.*

Proof. Let P_i be an orthogonal projection $(i = 1, 2)$. Hence $(\ker P_i)^{\perp} = M$ $(i = 1, 2)$. Then $P_1x - P_2x \in M$ and

$$\begin{aligned}\|P_1x - P_2x\|\|P_1x - P_2x\|_* &= [P_1x - P_2x, P_1x - P_2x]_{\varphi} = \\ &= [P_1x - x + x - P_2x, P_1x - P_2x]_{\varphi} = \\ &= [P_1x - x, P_1x - P_2x]_{\varphi} + [x - P_2x, P_1x - P_2x]\varphi = 0.\end{aligned}$$

Consequently, we conclude that $P_1x = P_2x$, which completes the proof. □

We can write the following statement as the result of our previous considerations.

Theorem 8. *Let X be a uniformly convex Banach space with continuous semi-inner product. Let M be a closed subspace of X. If $P : X \to M$ is a linear projection such that $\|P\| = 1$, then P is unique.*

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