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# FACTORIZATION FOR A CLASS OF TRIANGULAR MATRIX FUNCTIONS AND RELATED RIEMANN-HILBERT PROBLEMS

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Convolutions operators on a finite interval, which are important in many problems in engineering, physics and mathematics, are equivalent, in a sense, to Toeplitz operators of the form

$$T_G : (H_2^+)^2 \longrightarrow (H_2^+)^2$$
$$T_G\phi_+ = P^+(G\phi_+),$$

where the Hardy space $H_2^+$ , relative to the half-plane $\mathbb{C}^+$, is identified with a subspace of $L_2(\mathbb{R})$, and the symbol $G$ is a $2\times 2$ matrix function of the form ([1])

$$G(\xi) = \begin{bmatrix} e^{-i\lambda\xi} & 0 \\ g(\xi) & e^{i\lambda\xi} \end{bmatrix}, \quad \xi \in \mathbb{R}\,, \quad \lambda > 0, \quad g \in L_\infty(\mathbb{R}). \tag{1}$$

It is well-known that the Fredholm properties and invertibility of a Toeplitz operator can be completely studied if a Wiener-Hopf factorization of its symbol is known. In fact, the operator is Fredholm if and only if a Wiener-Hopf factorization of its symbol exists; and it is invertible if and only if this factorization is canonical. When the factors of such a canonical factorization are known explicitly, the inverse operator can be defined in terms of those factors ([6]).

When $G$ is an almost periodic (AP) function, if a Wiener-Hopf factorization exists then it must be canonical, and it coincides with a canonical APW factorization of $G$ when $G \in (APW)^{2\times 2}$(see [1], for instance). However, if a Wiener-Hopf factorization for G does not exist, we can still study the problem of APW factoriation of $G$, which appears as a natural generalization of a Wiener-Hopf factorization for this kind of matrix functions.

The existence and the actual determination of those factorizations are shown to be closely related to certain corona problems whose data are particular solutions to a Riemann-Hilbert problem

$$Gh_+ = h_-\,, \quad h_\pm \in H_\infty^\pm\,, \tag{2}$$

where $H_\infty^\pm$ denote the Hardy spaces consisting of the functions which are analytic and bounded in $\mathbb{C}^\pm$ (respectively), identified with subspaces of $L_\infty(\mathbb{R})$. We have:

**Theorem 1** ([3]). *Let $G \in (L_\infty(\mathbb{R}))^{2\times 2}$ and assume that the Riemann-Hilbert problem* (2) *admits a solution $(\phi_+, \phi_-)$ with $\phi_\pm \in CP^\pm$, where $CP^\pm = \{(\phi_{1\pm}, \phi_{2\pm}) \in (H_\infty^\pm) : \inf_{\mathbb{C}^\pm}(|\phi_{1\pm}| + |\phi_{2\pm}|) > 0\}$. Then $T_G$ is Fredholm if and only if $T_{\det G}$ is Fredholm, and in that case* $\operatorname{Ind} T_G = \operatorname{Ind} T_{\det G}$.

From this theorem we see that a non-trivial solution to a Riemann-Hilbert problem of the form (2) can provide important information to the study of the Fredholmness of $T_G$ and therefore of the Wiener-Hopf factorization of a matrix $G \in (L_\infty(\mathbb{R}))^{2\times 2}$. However, it seems natural to expect also important information from such a solution as regards $AP$ factorization when it is not canonical. For APW symbols this is indeed true and we can also obtain results on the AP partial indices and factors, as shown by the following.

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**Theorem 2** ([2]). *Let $G$ be an APW matrix function of the form* (1)*. Then $G$ admits an APW factorization if and only if the Riemann-Hilbert problem* (2) *admits a solution $\phi_\pm \in (APW^\pm)^2$ such that*

$$\tilde{\phi}_+ = e_{-\delta}\phi_+ \in CP^+ \quad \text{and} \quad \phi_- \in CP^- \qquad \text{for some} \quad \delta \geq 0.$$

*In this case, the AP partial indices of $G$ are $\pm\delta$ and we can take $\phi_-, \tilde{\phi}_+$ as the first columns of the factors of such a factorization.*

We apply the previous results to study a class of triangular matrix symbols $G$ of the form (1) with a non-diagonal entry of the form

$$g(\xi) = a_- e^{-i\beta\xi} + a_+ e^{i\nu\xi},$$

where $a_\pm \in H^\pm_\infty$ and $\nu, \beta \in ]0, \lambda[$. In this case, we show that a solution to the Riemann-Hilbert problem (2) can be determined explicitly and that conditions for existence of a Wiener-Hopf factorization of $G$ can be derived from them. We consider the cases $\nu + \beta \geq \lambda$ and $\nu + \beta < \lambda$ separately. First we study the case where $\nu + \beta \geq \lambda$, thus generalizing the results of [4] regarding the canonical factorization of this class of symbols, and secondly we obtain conditions for existence of a Wiener-Hopf factorization of $G$ when $\nu + \beta < \lambda$, see [2]. If in particular $a_\pm \in APW^\pm$, we establish conditions for the existence of an APW factorization of the symbol $G$ and, if it exists, we also determine its AP partial indices.

As mentioned above the determination of a solution to the Riemann-Hilbert problem (2) can be seen as a first step towards the explicit construction of a factorization for $G$. However, the study of the Riemann-Hilbert problem (2) has an interest that goes beyond that for at least two reasons: first, even if an AP factorization of $G$ does not exist, one still get some information on the solvability of (2); secondly, from an explicit solution of (2), it is sometimes possible to characterize new classes of matrix functions for which analogous conclusions can be obtained and thus widen the field of applications of the results.

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