9th Workshop Functional Analysis and its Applications, September 9–14, 2013, Nemecká, Slovakia, 42–43

ON A PERIODIC BOUNDARY VALUE PROBLEM AT RESONANCE

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The article deals with periodic boundary value problem for a second order differential equation with bounded nonlinearity. The associated linear problem is selfadjoint and the kernel of linear operator defined by this problem is two dimensional. The existence of a solution is based on a condition of Landesman - Lazer type [3]. The ideas of Ljapunov - Schmidt decomposition and Leray - Schauder degree are used [1], [7]. The case of one dimensional kernel for various boundary conditions was solved by Grossinho [2], Przeradzki [4] and in the earlier papers of the author [5, 6].

We consider the periodic boundary value problem

$$\begin{gathered} x'' + x + f(t,x) = h(t), \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{gathered} \tag{1}$$

where $I = [0, 2\pi]$, $f : I \times R \to R$ is a globally bounded continuous function and $h : I \to R$ is a a continuous function.

Solution $x(t)$ is a classical one, $x \in C^2(I)$.

The linear boundary value problem

$$\begin{gathered} x'' + x = h(t), \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{gathered} \tag{2}$$

is selfadjoint. By (2) is defined the linear operator $L : X \to Z$

$$Lx = x'' + x\,,$$

where $X = \{x \in C^2(I);\ x(0) = x(2\pi),\ x'(0) = x'(2\pi)\}$ and $Z = C(I)$.

The kernel and image of L are two dimensional, $N(L) = [\sin t, \cos t]$ and $\operatorname{Im} L = \{z \in Z;\ (z, \sin t) = (z, \cos t) = 0\}$, where $(.,.)$ is a scalar product in the space $L_2(I)$.

Let $N : X \to Z$ be the nonlinear operator defined by $N(x) = f(t, x(t))$.

The problem (1) rewritten in operator form is

$$Lx + N(x) = h\,. \tag{3}$$

Spaces X and Z are decomposed to direct sums $X = N(L) \oplus X_2$, $Z = Z_2 \oplus \operatorname{Im} L$. $\dim N(L) = \dim Z_2 = 2$. $N(L)$ and Z_2 are isomorphic with natural continuous isomorphism $J : Z_2 \to N(L)$.

The operator equation (3) is equivalent to the fixed point problem

$$x = T(x) \tag{4}$$

where $T : X \to X$ is defined as

$$T(x) = Px - JQ(N(x) - h) - L_p^{-1}(I - Q)(N(x) - h)\,. \tag{5}$$

Here P, Q are projections $P : X \to N(L)$, $Q : Z \to Z_2$ and $L_p^{-1} : \operatorname{Im} L \to X_2$ is the inverse operator to $L|_{X_2}$ continuous as the operator $\operatorname{Im} L \to X_2$ and compact as $Z \to Z$.

We prove the following existence theorem

Theorem. Suppose the existence of uniform limits

$$f_+(t) = \lim_{x\to\infty} f(t,x), \qquad f_-(t) = \lim_{x\to-\infty} f(t,x)\,.$$

If for each $t \in I$

$$f_+(t) > h(t) > f_-(t),$$

2010 *Mathematics Subject Classification.* 34C25.

Supported by grant 1/0426/12 of the Scientific Grant Agency VEGA of Slovak Republic.

then the periodic boundary value problem (1) is solvable.

Proof. The equation (4) is embedded to the homotopic system of equations

$$H(\lambda, x) = 0\,, \tag{6}$$

where $H : [0,1] \times Z \to Z$ is defined as $H(\lambda, x) = I - \lambda T(x)$.

The pair (λ, x) is a solution of (6) iff $x = x_1 + x_2 \in N(L) \oplus X_2$ and

$$\begin{aligned} x_2 + \lambda L_p^{-1}(I-Q)(N(x)-h) = 0, \\ (1-\lambda)x_1 + \lambda JQ(N(x)-h) = 0\,. \end{aligned} \tag{7}$$

A solution of (7) is for $\lambda = 1$ also solution of (4). The global boundedness of f implies boundedness of x_2 part of possible solution of (7). So exists a constant $R_2 > 0$ such that $||x_2|| < R_2$ for each solution x of (7). The bound R_2 is independent on λ.

We prove a priori boundedness of solution of (6) by contradiction. Suppose that (λ_n, x_n) is a sequence of solutions of (6) such that $||x_n|| \to \infty$. As $||x_{n2}|| < R_2$, $||x_{n1}|| \to \infty$. Going to the subsequence we obtain that $x_{n1} = A_n \sin(t - \varphi_n)$ with $A_n \to \infty$ and $\varphi_n \to \varphi$. Then

$$x_{n1} = -\frac{\lambda}{1-\lambda} JQ(N(x)-h)$$

and
$$\int_0^{2\pi} x_{1n} \sin(t-\varphi_n)\,dt = -\frac{\lambda}{1-\lambda}\int_0^{2\pi} Q(N(x)-h)\sin(t-\varphi_n)\,dt =$$

$$= -\frac{\lambda}{1-\lambda}\int_0^{2\pi} (f(t, A_n \sin(t-\varphi_n) + x_{n2}(t)) - h(t))\sin(t-\varphi_n)\,dt\,. \tag{8}$$

Assuming $0 < \lambda < 1$, we obtain for $n \to \infty$

$$0 > \frac{\lambda - 1}{\lambda} A_n \pi = \int_{I_\varphi^+} (f_+(t) - h(t))\sin(t-\varphi)\,dt + \int_{I_\varphi^-} (f_-(t) - h(t))\sin(t-\varphi)\,dt\,, \tag{9}$$

where $I_\varphi^+ = \{t \in I; \sin(t-\varphi) > 0\}$ and $I_\varphi^- = \{t \in I; \sin(t-\varphi) < 0\}$. The last inequality (8) is in a contradiction with assumptions of theorem. That means solutions of (6) are bounded for $0 < \lambda < 1$. Denote the a priori bound by R. For $\lambda = 0$ is the left hand side of (6) identity with only zero solution.

Set $\Omega = \{x \in Z;\ ||x|| < R\}$. Now either (6) possesses a solution for $\lambda = 1$ on the set $\partial\Omega$ or the Leray - Schauder degree of H is well defined on Ω for each $0 \le \lambda \le 1$ and

$$d(I - \lambda T, \Omega, 0) = d(I, \Omega, 0) = 1.$$

Then (4) possesses a solution $x(t)$ which is a classical solution of the periodic boundary value problem (1).

□

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