# CHARACTERIZATION OF STABILITY OF CONTRACTIONS

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Abstract. This talk is based on a joint paper with L. Kérchy. We characterize those sequences $\{h_n\}_{n=1}^{\infty}$ of bounded analytic functions, which have the property that an absolutely continuous contraction $T$ is stable (that is the powers $T^n$ converge to zero) exactly when the operators $h_n(T)$ converge to zero in the strong operator topology. Our result is extended to polynomially bounded operators too.

## 1. Introduction

Let $\mathcal{H}$ be a complex, separable Hilbert space, and let $\mathcal{L}(\mathcal{H})$ stand for the algebra of bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is a contraction if $||T|| \leq 1$. It is well-known that any contraction can be uniquely decomposed into the orthogonal sum $T = T_1 \oplus T_2$ of a completely nonunitary (c.n.u.) contraction $T_1$ and a unitary operator $T_2$. A contraction is called absolutely continuous if the scalar-valued spectral measure of its unitary part is absolutely continuous with respect to Lebesgue measure. The Sz.-Nagy–Foias functional calculus $\phi_T$ for an a.c. contraction $T$ is a contractive, weak-* continuous, unital algebra homomorphism from the Hardy space $H^\infty$ into $\mathcal{L}(\mathcal{H})$, mapping the identity function $\chi$ into $T$.

The contraction $T \in \mathcal{L}(\mathcal{H})$ is called stable, in notation $T \in C_{0\cdot}$, if $T^n$ converges to the zero operator in strong operator topology (SOT). Our aim in this note is to characterize those sequences $\{h_n\}_{n=1}^{\infty} \subset H^\infty$, which can serve to test stability of an a.c. contraction, namely, satisfying the condition that $h_n(T) \to 0$ (SOT) exactly when $T \in C_{0\cdot}$. This question was posed by M. Dritschel.

## 2. Main result

**Definition 1.** A sequence of bounded analytic functions $\{h_n\}_{n=1}^{\infty} \subset H^\infty$ is a *test sequence of stability for a.c. contractions* if for every a.c. contraction $T$ the condition $T^n \to 0$ (SOT) holds exactly when $h_n(T) \to 0$ (SOT).

**Theorem 2.** *A sequence of bounded analytic functions $\{h_n\}_{n=1}^{\infty} \subset H^\infty$ is a test sequence of stability for a.c. contractions if and only if*

*(i)* $\lim_{n\to\infty} h_n(z) = 0$ *for all* $z \in \mathbb{D}$,

*(ii)* $\sup\{||h_n||_\infty : n \in \mathbb{N}\} < \infty$,

*(iii)* $\limsup_{n\to\infty} ||\chi_\alpha h_n||_2 > 0$ *for every Borel set* $\alpha \subset \mathbb{T}$ *of positive measure.*
*($\chi_\alpha$ is the characteristic function of $\alpha$.)*

Besides this theorem some connected results and extensions to polynomially bounded operators were communicated in [4].

## References

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