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# IDEALS WITH AT MOST COUNTABLE HULL IN CERTAIN ALGEBRAS OF FUNCTIONS ANALYTIC ON THE HALF-PLANE

ANDRZEJ SOŁTYSIAK

ABSTRACT. We describe all closed ideals with at most countable hull in the algebras $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$ $(\alpha > 0)$ of analytic functions on the complex half-plane.

The space $\mathcal{A}^{(n)}(\mathbb{C}^+)$ $(n \in \mathbb{N})$ is the set of functions $F$ analytic on the right half-plane $\mathbb{C}^+ = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$ continuously extendable to $i\mathbb{R}$, whose derivatives $F^{(k)}$ are continuous on $\overline{\mathbb{C}^+} \setminus \{0\}$ and satisfy $\lim_{z\to 0} z^k F^{(k)}(z) = 0$ for $1 \le k \le n$, while $\lim_{z\to\infty} z^k F^{(k)}(z) = 0$, $0 \le k \le n$ (we denote $F^{(0)} = F$).

For a bounded function $F$ on $\mathbb{C}^+$ let $\|F\|_\infty = \sup_{z\in\mathbb{C}^+} |F(z)|$. Provided with the norm $\|F\|_{(n)} = \sum_{j=0}^{n} \|\zeta^j F^{(j)}\|_\infty$ ($\zeta$ stands for the identity function $z \mapsto z$) and the pointwise multiplication the space $\mathcal{A}^{(n)}(\mathbb{C}^+)$ is a Banach algebra. Notice that the norm $\|F\|_{(n)}$ is equivalent to the norm $\|F\|_n = \|F\|_\infty + \|\zeta^n F^{(n)}\|_\infty$ (see [2], Prop. 3.3 and Rem. 3.6).

The space $\mathcal{A}^{(0)}(\mathbb{C}^+)$ is the set of continuous functions on $\overline{\mathbb{C}^+}$ vanishing at infinity and analytic on the half-plane $\mathbb{C}^+$. It is a Banach algebra with the pointwise multiplication and the norm $\|F\|_\infty = \sup_{z\in\mathbb{C}^+} |F(z)|$.

The spaces $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$, $\alpha > 0$, are defined by means of the fractional complex derivation introduced in [2].

For $F \in \mathcal{A}^{(n)}(\mathbb{C}^+)$, $\alpha > 0$, $n = [\alpha] + 1$, and $z = re^{i\theta} \in \overline{\mathbb{C}^+}$ the complex $\alpha$-derivative of $F$ is given by the formula

$$W^\alpha F(z) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_0^\infty t^{n-\alpha-1} F^{(n)}(z + te^{i\theta})\, dt.$$

It should be mentioned that the integral in the above formula is independent of $\theta$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ ([2], Lemma 3.1).

For an arbitrary $F \in \mathcal{A}^{(n)}(\mathbb{C}^+)$ let $\|F\|_\alpha = \|F\|_\infty + \sup_{z\in\mathbb{C}^+} |z|^\alpha |W^\alpha F(z)|$.

The space $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$ is defined as the completion of the space $\mathcal{A}^{(n)}(\mathbb{C}^+)$ in the norm $\|\cdot\|_\alpha$ $(n = [\alpha] + 1)$.

Propositions 3.5 and 3.8 from [2] provide the following properties of the family of spaces $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$, $\alpha > 0$.

**Theorem 1.** [2] (i) *For every $\alpha > 0$ the space $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$ is a Banach algebra under the pointwise multiplication.*

(ii) *For $\beta \ge \alpha \ge 0$ there is a constant $C_{\alpha\beta} > 0$ such that $\|F\|_\alpha \le C_{\alpha\beta} \|F\|_\beta$.*

*Consequently, there is a natural continuous embedding $\mathcal{A}^{(\beta)}(\mathbb{C}^+) \hookrightarrow \mathcal{A}^{(\alpha)}(\mathbb{C}^+)$ with a dense range.*

The family of algebras $\mathcal{A}^{(n)}(\mathbb{C}^+)$ $(n \in \mathbb{N}_0)$ as well as the algebras $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$ for $\alpha > 0$ appeared in the paper [2] as the spaces of Gelfand transforms of "fractional convolution algebras" of functions on the half-line $\mathbb{R}^+ = (0, +\infty)$.

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The talk is based on the joint work [5] with Antoni Wawrzyńczyk from Universidad Autónoma Metropolitana-Iztapalapa in México City.

Closed ideals of the algebra $\mathcal{A}^{(n)}(\mathbb{C}^+)$ were described in [3]. In the present paper we study the ideals of the algebras $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$ $(\alpha > 0)$.

Let $m(w) = \dfrac{1+w}{1-w}$ be the Möbius transform. It carries the disc $\mathbb{D}$ onto the half-plane $\mathbb{C}^+$ and the circle $\mathbb{T}$ onto $i\mathbb{R} \cup \{\infty\}$. The inverse of $m$ is the function $z \mapsto \dfrac{z-1}{z+1}$. If $U$ is an inner function on the unit disc $\mathbb{D}$, then the function $\mathcal{U} = U \circ m^{-1}$ on $\mathbb{C}^+$ can be represented as a product $U \circ m^{-1} = BS$, where $B$ is of the form

$$B(z) = \left(\frac{z-1}{z+1}\right)^k \prod_n \frac{|1-z_n^2|}{1-z_n^2} \cdot \frac{z-z_n}{z+\overline{z}_n},$$

while $S$ is uniquely representable as

$$S(z) = e^{-\rho z} \exp\left(-\int_{\mathbb{R}} \frac{tz+i}{t+iz}\, d\mu(t)\right)$$

for some positive measure $\mu$ on $\mathbb{R}$ singular with respect to the Lebesgue measure and $\rho \geq 0$. Hence, the set of zeros of the function $B$ coincides with the set of zeros of $U \circ m^{-1}$ in the open half-plane $\mathbb{C}^+$ and the set of zeros of the factor $S$ consists of zeros of $U \circ m^{-1}$ in the imaginary axis $i\mathbb{R}$ and is equal to the support of $\mu$. Functions on $\mathbb{C}^+$ of the form $\mathcal{U} = BS$ are called inner functions on the half-plane.

Now, let $\mathcal{U}$ be an inner functions on the half-plane. Let $\mathcal{H} = \{H^j\}_{j=0}^{[\alpha]}$ be an arbitrary descending family of closed subsets of $i\mathbb{R}$ such that $0 \notin H^j$ for $j > 0$. Let us denote

$$\mathcal{I}(\mathcal{U};\mathcal{H}) = \{F \in \mathcal{A}^{(\alpha)}(\mathbb{C}^+)\colon\ \mathcal{U} \,|\, F \text{ and } F^{(j)} \text{ vanishes on } H^j \text{ for } 0 \leq j \leq [\alpha]\,\}.$$

If $\mathcal{I}$ is a closed ideal in $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$, then we denote

$$H^0(\mathcal{I}) = \{z \in i\mathbb{R}\colon\ F(z) = 0 \text{ for all } F \in \mathcal{I}\,\},$$

and for $0 < j \leq [\alpha]$

$$H^j(\mathcal{I}) = \{z \in i\mathbb{R} \setminus \{0\}\colon\ F(z) = F'(z) = \ldots = F^{(j)}(z) = 0 \text{ for all } F \in \mathcal{I}\,\}.$$

**Theorem 2.** *Let $\mathcal{I}$ be a closed ideal in $\mathcal{A}^{(\alpha)}(\mathbb{C}^+)$ such that $H^0(\mathcal{I})$ is at most countable. Let $\mathcal{U}_{\mathcal{I}}$ be the greatest common inner divisor of all elements of $\mathcal{I}$. Then*

$$\mathcal{I} = \mathcal{I}(\mathcal{U}_{\mathcal{I}}; H^0(\mathcal{I}), \ldots, H^{[\alpha]}(\mathcal{I})).$$

The proof of this result is based upon the main theorem of [4](Thm. 5.3) (see also Proceedings of the 8th WFA, September 5–10, 2011, Nemecká) which gives the description of the closed ideals with at most countable hull of certain subalgebras of the disc algebra.

## References


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Faculty of Mathematics and Computer Science, Adam Mickiewicz University, ul. Umultowska 87, 61–614 Poznań, Poland

*E-mail address*: asoltys@amu.edu.pl