On the reflexivity and transitivity of the Toeplitz operators on the upper half-plane

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Joint work with M. Ptak

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Let $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$, $\mathbb{T} = \{\omega \in \mathbb{C} : |\omega| = 1\}$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

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Definition 1

The Hardy space $H^p(\mathbb{C}_+)$ $(0 < p < \infty)$ on $\mathbb{C}_+$ is the space of all analytic functions $F \colon \mathbb{C}_+ \to \mathbb{C}$ such that

$$\|F\|_{H^p(\mathbb{C}_+)} := \sup_{y>0} \Big(\int_{-\infty}^{\infty} |F(x+iy)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

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$H^\infty(\mathbb{C}_+)$ is the space of all bounded and analytic functions on $\mathbb{C}_+$ with $\|F\|_{H^\infty(\mathbb{C}_+)} = \sup_{y>0} |F(x+iy)|$.

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$L^p(\mathbb{T}) := L^p([0, 2\pi], dm)$, $L^p(\mathbb{R}) := L^p(\mathbb{R}, dx)$ Banach spaces ($p \geqslant 1$).

At first we recall well known an isomorphism between the space $L^p(\mathbb{T})$ and $L^p(\mathbb{R})$.

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Lemma 2 (Nikolski, *Operators*, ...)

The mapping

$$(U_p f)(t) = \left(\frac{1}{\pi(t+i)^2}\right)^{1/p} f(\gamma(t)), \ t \in \mathbb{R} \tag{1}$$

is an isometric isomorphism of the space $L^p(\mathbb{T})$ *onto* $L^p(\mathbb{R})$ *for* $1 \leqslant p < \infty$.

Lemma 3

An operator $U_\infty \colon L^\infty(\mathbb{T}) \to L^\infty(\mathbb{R})$ *defined by*

$$U_\infty \varphi = \varphi \circ \gamma \tag{2}$$

is an isometric isomorphism.

We will use the duality between $L^1(\mathbb{T})$ and $L^\infty(\mathbb{T})$ and also between $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ hence we have to define the isomorphism between $L^1(\mathbb{T})$ and $L^1(\mathbb{R})$ differently than (1) of lemma 2.

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Lemma 4

An operator $U_1 \colon L^1(\mathbb{T}) \to L^1(\mathbb{R})$ defined by

$$(U_1 f)(t) = \frac{1}{\pi}\frac{1}{1+t^2}\, f(\gamma(t)) \tag{3}$$

is an isometric isomorphism.

The above definition of U_1 let see U_∞ given by (2) of lemma 3 as a dual action to $(U_1)^{-1}\colon L^1(\mathbb{R}) \to L^1(\mathbb{T})$.

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Theorem 5 (WM, Ptak)

Let $U_\infty\colon L^\infty(\mathbb{T}) \to L^\infty(\mathbb{R})$ be given by $U_\infty\varphi = \varphi \circ \gamma$ and $U_1\colon L^1(\mathbb{T}) \to L^1(\mathbb{R})$ given by $(U_1 f)(t) = \frac{1}{\pi}\frac{1}{1+t^2} f(\gamma(t))$, then

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(a) $\langle \varphi, f\rangle = \langle U_\infty \varphi, U_1 f\rangle$ *for all* $\varphi \in L^\infty(\mathbb{T})$, $f \in L^1(\mathbb{T})$.

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(a) $\langle \varphi, f\rangle = \langle U_\infty\varphi, U_1 f\rangle$ *for all* $\varphi \in L^\infty(\mathbb{T})$, $f \in L^1(\mathbb{T})$.

(b) $U_\infty = (U_1^{-1})^*$.

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(a) $\langle \varphi, f\rangle = \langle U_\infty\varphi, U_1 f\rangle$ *for all* $\varphi \in L^\infty(\mathbb{T})$, $f \in L^1(\mathbb{T})$.

(b) $U_\infty = (U_1^{-1})^*$.

(c) U_∞ *is a weak* homeomorphism.*

Lemma 6

If $\varphi \in L^\infty(\mathbb{T})$ and M_φ be a multiplication operator on the space $L^2(\mathbb{T})$ then

$$U_2 M_\varphi U_2^{-1} = M_{\varphi\circ\gamma}.$$

Now, we identify spaces $H^p(\mathbb{D})$ with $H^p(\mathbb{C}_+)$.

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Lemma 7 (Nikolski, *Operators*, ...)

The mapping

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is an isometric isomorphism of the space $H^p(\mathbb{D})$ *onto* $H^p(\mathbb{C}_+)$ *for* $1 \leqslant p < \infty$.

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The mapping

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Lemma 8

An operator $U_\infty \colon H^\infty(\mathbb{D}) \to H^\infty(\mathbb{C}_+)$ given by $U_\infty g = g \circ \gamma, \; \gamma \in H^\infty(\mathbb{D})$ is an isometric isomorphism.

Definition 9

For each $\varphi \in L^\infty(\mathbb{T})$ ($\Phi \in L^\infty(\mathbb{R})$) a Toeplitz operator on $H^2(\mathbb{D})$ ($H^2(\mathbb{C}_+)$) with symbol φ (Φ) is an operator T_φ (T_Φ) defined by

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$$T_\varphi f = P_{H^2(\mathbb{D})}(\varphi f),\ f \in H^2(\mathbb{D})$$
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where $P_{H^2(\mathbb{D})}$ ($P_{H^2(\mathbb{C}_+)}$) is the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$ ($L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$).

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If $\varphi \in H^\infty(\mathbb{D})$ ($\Phi \in H^\infty(\mathbb{C}_+)$) then T_φ (T_Φ) is called analytic Toeplitz operator.

By $\mathcal{T}(\mathbb{D})$ ($\mathcal{T}(\mathbb{C}_+)$) we denote the space of all Toeplitz operators and by $\mathcal{A}(\mathbb{D})$ ($\mathcal{A}(\mathbb{C}_+)$) the algebra of all analytic Toeplitz operators on $H^2(\mathbb{D})$ ($H^2(\mathbb{C}_+)$).

Let $\mathcal{H}$ be a Hilbert space. By $\mathcal{B}(\mathcal{H})$ we denote the set of all linear and bounded operators on $\mathcal{H}$ and by $\mathcal{B}_1(\mathcal{H})$ the set of all trace-class operators on $\mathcal{H}$.

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$\xi \colon L^\infty(\mathbb{T}) \to \mathcal{B}(H^2(\mathbb{D}))$ ($\eta \colon L^\infty(\mathbb{R}) \to \mathcal{B}(H^2(\mathbb{C}_+))$) given by $\xi(\varphi) = T_\varphi$ ($\eta(\Phi) = T_\Phi$) is a symbol map of the Toeplitz operator on $H^2(\mathbb{D})$ ($H^2(\mathbb{C}_+)$).

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The relationship between the Toeplitz operators on $H^2(\mathbb{D})$ and $H^2(\mathbb{C}_+)$ is characterized as follows.

Theorem 10

If $\widetilde{U}_2 : \mathcal{B}(H^2(\mathbb{D})) \to \mathcal{B}(H^2(\mathbb{C}_+))$ *is given by* $\widetilde{U}_2(A) = U_2 A U_2^{-1}$ *then*

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(a) $U_2 T_\varphi U_2^{-1} = T_{\varphi\circ\gamma}$, $\varphi \in L^\infty(\mathbb{T})$.

(b) $U_2(\mathcal{T}(\mathbb{D}))U_2^{-1} = \mathcal{T}(\mathbb{C}_+)$, $U_2(\mathcal{A}(\mathbb{D}))U_2^{-1} = \mathcal{A}(\mathbb{C}_+)$.

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(d) *The following diagram commutes*

$$\begin{array}{ccc} L^\infty(\mathbb{T}) & \xrightarrow{\ \xi\ } & \mathcal{T}(\mathbb{D}) \\ {\scriptstyle U_\infty}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{U}_2} \\ L^\infty(\mathbb{R}) & \xrightarrow[\ \eta\ ]{} & \mathcal{T}(\mathbb{C}_+) \end{array}$$

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If $U\colon X \to Y$ is linear then an operator $U_*\colon Y_* \to X_*$ is defined by the following formula

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$\mathcal{S}_\perp$ is the preannihilator of $\mathcal{S} \subset X$.

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The relationship between this spaces is given by the following Theorem.

Theorem 11 (WM, Ptak)

If $\widetilde{U_2}$ *is given by* $\widetilde{U_2}(A) = U_2 A U_2^{-1}$, $A \in \mathcal{B}(H^2(\mathbb{D}))$ *and* U_1 *is given by* $(U_1 f)(t) = \frac{1}{\pi}\frac{1}{1+t^2} f(\gamma(t))$, $f \in L^1(\mathbb{T})$, *then*

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(a) $\langle T_\varphi, \xi_*^{-1}(f)\rangle = \langle T_{U_\infty \varphi}, \eta_*^{-1}(U_1 f)\rangle$ *for all* $\varphi \in L^\infty(\mathbb{T})$, $f \in L^1(\mathbb{T})$.

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(b) *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{T}(\mathbb{C}_+)_* & \xrightarrow{\eta_*} & L^1(\mathbb{R}) \\ {\scriptstyle \widetilde{U}_{2*}}\downarrow & & \downarrow{\scriptstyle U_1^{-1}} \\ \mathcal{T}(\mathbb{D})_* & \xrightarrow[\xi_*]{} & L^1(\mathbb{T}) \end{array}$$

Definition 12

The *reflexive closure* of a subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ is given by

$$\operatorname{ref}\mathcal{S} = \{B \in \mathcal{B}(\mathcal{H}) : Bh \in \overline{\mathcal{S}h} \ \text{ for all } h \in \mathcal{H}\}.$$

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$\mathcal{S}$ is said to be *reflexive* if $\operatorname{ref}\mathcal{S} = \mathcal{S}$ and *transitive* if $\operatorname{ref}\mathcal{S} = \mathcal{B}(\mathcal{H})$.

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(1) *$\mathcal{B}$ is not transitive.*

(2) *There is a function $f : \mathbb{T} \to \mathbb{C}$ such that $f \in L^1(\mathbb{T})$, $\log|f| \in L^1(\mathbb{T})$ and $\int_{\mathbb{T}} \varphi f dm = 0$ for all $T_\varphi \in \mathcal{B}$.*

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(3) *$\mathcal{B}$ is reflexive.*

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(1) *$\mathcal{F}$ is not transitive.*

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Suppose that $\mathcal{F} \subset \mathcal{T}(\mathbb{C}_+)$ is a weak* closed. Then the following statements are equivalent.

(1) *$\mathcal{F}$ is not transitive.*

(2) *There is a function $F\colon \mathbb{R} \to \mathbb{C}$ such that $F \in L^1(\mathbb{R})$, $\log|F| \in L^1\left(\mathbb{R}, \frac{dt}{1+t^2}\right)$ and $\int_{\mathbb{R}} \Phi F dt = 0$ for all $T_\Phi \in \mathcal{F}$.*

(3) *$\mathcal{F}$ is reflexive.*

- $\mathcal{A}(\mathbb{C}_+)$ is reflexive.

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- If $\overline{F}$ is inner function on $\mathbb{C}_+$ then $T_F\mathcal{A}(\mathbb{C}_+)$ is reflexive.

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- If $\overline{F}$ is inner function on $\mathbb{C}_+$ then $T_F\mathcal{A}(\mathbb{C}_+)$ is reflexive.
- If $F \in L^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \big|\log|F(t)|\big| \frac{dt}{1+t^2} = \infty$ then $T_F\mathcal{A}(\mathbb{C}_+)$ is transitive.

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