Decompositions of contractions and power bounded operators

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So $A_n(T) \to P \quad (SOT)$, where P is a projection onto $N(T - I)$

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If T is a Hilbert space contraction, then the spaces Y_1, Z_1 are orthogonal.

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$$x \in Z_2 \Leftrightarrow \lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} |\langle T^j x, x^* \rangle| = 0 \text{ for all } x^* \in X^*$$

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Then Y_3, Z_3 *are* T*-invariant subspaces and* $H = Y_3 \oplus Z_3$ *(orthogonal sum).*

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Let $T \in B(H)$ be a Hilbert space contraction. Let Y_4 and Z_4 be the sets of all $x \in H$ such that there exists a singular (absolutely continuous) measure μ_x with

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for all polynomials p. Then Y_4, Z_4 are orthogonal T-invariant subspaces and $H = Y_3 \oplus Z_3$.

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Let $T \in B(H)$ *be a Hilbert space contraction. Then there are orthogonal* T*-invariant subspaces* $Y_5, Z_5 \subset H$ *such that* $T|Y_5$ *is unitary and* $T|Z_5$ *completely non-unitary.*