Reflexivity defect of the kernel of some elementary operators

Tina Rudolf

University of Ljubljana

September 2011

k-reflexive cover

- $\mathcal{X}$ finite dimensional complex Banach space,
- $L(\mathcal{X})$ all linear operators and $\mathcal{S} \subseteq L(\mathcal{X})$ non-empty,
- k positive integer.

k-reflexive cover

- $\mathcal{X}$ finite dimensional complex Banach space,
- $L(\mathcal{X})$ all linear operators and $\mathcal{S} \subseteq L(\mathcal{X})$ non-empty,
- k positive integer.

Definition

k-reflexive cover of $\mathcal{S}$

$$\mathrm{Ref_k}(\mathcal{S}) = \{T \in L(\mathcal{X}) : \forall \varepsilon > 0, \ \forall x_1, \ldots, x_k \in \mathcal{X} : \ \exists S \in \mathcal{S} : \\ \|Tx_i - Sx_i\| < \varepsilon \qquad \forall i \in \{1, \ldots, k\}\}.$$

k-reflexive cover

- $\mathcal{X}$ finite dimensional complex Banach space,
- $L(\mathcal{X})$ all linear operators and $\mathcal{S} \subseteq L(\mathcal{X})$ non-empty,
- k positive integer.

Definition

k-reflexive cover of $\mathcal{S}$

$$\mathrm{Ref_k}(\mathcal{S}) = \{T \in L(\mathcal{X}) : \forall\, \varepsilon > 0,\ \forall\, x_1, \ldots, x_k \in \mathcal{X} :\ \exists S \in \mathcal{S} : \\ \|Tx_i - Sx_i\| < \varepsilon \qquad \forall\, i \in \{1, \ldots, k\}\}.$$

Linear subspace $\mathcal{S} \subset L(\mathcal{X})$ is k-reflexive if $\mathrm{Ref_k}(\mathcal{S}) = \mathcal{S}$.

k-reflexivity defect and its properties

Definition

The *k-reflexivity defect* of $\mathcal{S}$

$$\mathrm{rd}_k(\mathcal{S}) = \dim(\mathrm{Ref_k}(\mathcal{S})/\mathcal{S}).$$

k-reflexivity defect and its properties

Definition

The *k-reflexivity defect* of $\mathcal{S}$

$$\mathrm{rd}_k(\mathcal{S}) = \dim(\mathrm{Ref_k}(\mathcal{S})/\mathcal{S}).$$

- $\mathcal{S}$ k-reflexive $\iff$ $\mathrm{rd}_k(\mathcal{S}) = 0$,

k-reflexivity defect and its properties

Definition

The *k-reflexivity defect* of $\mathcal{S}$

$$\mathrm{rd}_k(\mathcal{S}) = \dim(\mathrm{Ref_k}(\mathcal{S})/\mathcal{S}).$$

- $\mathcal{S}$ k-reflexive $\iff$ $\mathrm{rd}_k(\mathcal{S}) = 0$,
- $\mathrm{rd}_k(\mathcal{S}) \geq \mathrm{rd}_{k+1}(\mathcal{S})$,

k-reflexivity defect and its properties

Definition

The *k-reflexivity defect* of $\mathcal{S}$

$$\mathrm{rd}_k(\mathcal{S}) = \dim(\mathrm{Ref}_k(\mathcal{S})/\mathcal{S}).$$

- $\mathcal{S}$ k-reflexive $\iff$ $\mathrm{rd}_k(\mathcal{S}) = 0$,
- $\mathrm{rd}_k(\mathcal{S}) \geq \mathrm{rd}_{k+1}(\mathcal{S})$,
- $S^{(k)} = S \oplus \cdots \oplus S, \quad \mathcal{S}^{(k)} := \{S^{(k)} : S \in \mathcal{S}\}$,

k-reflexivity defect and its properties

Definition

The k-reflexivity defect of $\mathcal{S}$

$$\mathrm{rd}_k(\mathcal{S}) = \dim(\mathrm{Ref_k}(\mathcal{S})/\mathcal{S}).$$

- $\mathcal{S}$ k-reflexive $\iff$ $\mathrm{rd}_k(\mathcal{S}) = 0$,
- $\mathrm{rd}_k(\mathcal{S}) \geq \mathrm{rd}_{k+1}(\mathcal{S})$,
- $S^{(k)} = S \oplus \cdots \oplus S$, $\quad \mathcal{S}^{(k)} := \{S^{(k)} : \; S \in \mathcal{S}\}$,
- $\mathcal{S}$ k-reflexive $\iff$ $\mathcal{S}^{(k)}$ reflexive,
- $\mathrm{rd}_k(\mathcal{S}) = \mathrm{rd}\big(\mathcal{S}^{(k)}\big)$.

Properties of k-reflexivity defect

- $\mathrm{rd}_k(\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_n) = \mathrm{rd}_k(\mathcal{S}_1) + \ldots + \mathrm{rd}_k(\mathcal{S}_n)$,

Properties of *k*-reflexivity defect

- $\mathrm{rd}_k(\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_n) = \mathrm{rd}_k(\mathcal{S}_1) + \ldots + \mathrm{rd}_k(\mathcal{S}_n)$,
- A and B invertible $\quad \Rightarrow \quad \mathrm{rd}_k(B\,\mathcal{S}\,A) = \mathrm{rd}_k(\mathcal{S})$.

Properties of k-reflexivity defect

- $\mathrm{rd}_k(\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_n) = \mathrm{rd}_k(\mathcal{S}_1) + \ldots + \mathrm{rd}_k(\mathcal{S}_n)$,
- A and B invertible $\quad \Rightarrow \quad \mathrm{rd}_k(B\,\mathcal{S}\,A) = \mathrm{rd}_k(\mathcal{S})$.

Proposition

- $\mathcal{X} = \mathcal{X}_1 \oplus \ldots \oplus \mathcal{X}_N, \quad k \in \mathbb{N}$,
- $\mathcal{S} = [\mathcal{S}_{ij}] \subseteq L(\mathcal{X})$ *linear subspace.*

Properties of k-reflexivity defect

- $\mathrm{rd}_k(\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_n) = \mathrm{rd}_k(\mathcal{S}_1) + \ldots + \mathrm{rd}_k(\mathcal{S}_n)$,
- A and B invertible $\quad \Rightarrow \quad \mathrm{rd}_k(B\,\mathcal{S}\,A) = \mathrm{rd}_k(\mathcal{S})$.

Proposition

- $\mathcal{X} = \mathcal{X}_1 \oplus \ldots \oplus \mathcal{X}_N, \quad k \in \mathbb{N}$,
- $\mathcal{S} = [\mathcal{S}_{ij}] \subseteq L(\mathcal{X})$ *linear subspace.*

Then:

(i) $\mathrm{Ref_k}(\mathcal{S}) = [\mathrm{Ref_k}(\mathcal{S}_{ij})]$,

Properties of k-reflexivity defect

- $\mathrm{rd}_k(\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_n) = \mathrm{rd}_k(\mathcal{S}_1) + \ldots + \mathrm{rd}_k(\mathcal{S}_n)$,
- A and B invertible $\quad \Rightarrow \quad \mathrm{rd}_k(B\,\mathcal{S}\,A) = \mathrm{rd}_k(\mathcal{S})$.

Proposition

- $\mathcal{X} = \mathcal{X}_1 \oplus \ldots \oplus \mathcal{X}_N, \quad k \in \mathbb{N}$,
- $\mathcal{S} = [\mathcal{S}_{ij}] \subseteq L(\mathcal{X})$ *linear subspace.*

Then:

(i) $\mathrm{Ref_k}(\mathcal{S}) \;=\; [\mathrm{Ref_k}(\mathcal{S}_{ij})]$,

(ii) $\mathrm{rd}_k(\mathcal{S}) = \sum_{i=1}^{M} \sum_{j=1}^{N} \mathrm{rd}_k(\mathcal{S}_{ij})$,

Properties of k-reflexivity defect

- $\mathrm{rd}_k(\mathcal{S}_1 \oplus \ldots \oplus \mathcal{S}_n) = \mathrm{rd}_k(\mathcal{S}_1) + \ldots + \mathrm{rd}_k(\mathcal{S}_n)$,
- A and B invertible $\quad \Rightarrow \quad \mathrm{rd}_k(B\,\mathcal{S}\,A) = \mathrm{rd}_k(\mathcal{S})$.

Proposition

- $\mathcal{X} = \mathcal{X}_1 \oplus \ldots \oplus \mathcal{X}_N, \quad k \in \mathbb{N}$,
- $\mathcal{S} = [\mathcal{S}_{ij}] \subseteq L(\mathcal{X})$ *linear subspace.*

Then:

(i) $\mathrm{Ref_k}(\mathcal{S}) = [\mathrm{Ref_k}(\mathcal{S}_{ij})]$,

(ii) $\mathrm{rd}_k(\mathcal{S}) = \sum_{i=1}^{M} \sum_{j=1}^{N} \mathrm{rd}_k(\mathcal{S}_{ij})$,

(iii) $\mathcal{S}$ *is reflexive* $\iff$ $\mathcal{S}_{ij}$ *is reflexive* $\forall\, i, j$.

Elementary operators

- $(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$

Elementary operators

- $(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$

Definition

Elementary operator with coefficients A_i, B_j is

$$\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k \qquad (T \in L(\mathcal{X})).$$

Elementary operators

- $(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$

Definition

Elementary operator with coefficients A_i, B_j is

$$\Delta(T) = B_1 TA_1 + B_2 TA_2 + \ldots + B_k TA_k \qquad (T \in L(\mathcal{X})).$$

- $\ker \Delta$ is k-reflexive, i.e., $\mathrm{rd}_k(\ker \Delta) = 0$,

Elementary operators

- $(A_1, \ldots, A_k)$, $(B_1, \ldots, B_k)$ $\quad$ k-tuples of operators on $\mathcal{X}$

Definition

Elementary operator with coefficients A_i, B_j is

$$\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k \qquad (T \in L(\mathcal{X})).$$

- $\ker \Delta$ is k-reflexive, i.e., $\mathrm{rd}_k(\ker \Delta) = 0$,
- $j < k$: $\quad \mathrm{rd}_j(\ker \Delta) = ?$

Elementary operators of length 2

Elementary operator of length 2 with coefficients A_1, A_2, B_1, B_2

$$\Delta(T) = B_1 T A_1 - B_2 T A_2, \qquad T \in L(\mathcal{X}).$$

Elementary operators of length 2

Elementary operator of length 2 with coefficients A_1, A_2, B_1, B_2

$$\Delta(T) = B_1 T A_1 - B_2 T A_2, \qquad T \in L(\mathcal{X}).$$

- $\mathcal{X} \equiv \mathbb{C}^n$,

Elementary operators of length 2

Elementary operator of length 2 with coefficients A_1, A_2, B_1, B_2

$$\Delta(T) = B_1 T A_1 - B_2 T A_2, \qquad T \in L(\mathcal{X}).$$

- $\mathcal{X} \equiv \mathbb{C}^n$,
- $L(\mathcal{X}) \equiv \mathbb{M}_n$ the algebra of n-by-n matrices.

Elementary operators of length 2

Elementary operator of length 2 with coefficients A_1, A_2, B_1, B_2

$$\Delta(T) = B_1 T A_1 - B_2 T A_2, \qquad T \in L(\mathcal{X}).$$

- $\mathcal{X} \equiv \mathbb{C}^n$,
- $L(\mathcal{X}) \equiv \mathbb{M}_n$ the algebra of n-by-n matrices.

Question

How much is the reflexivity defect of $\ker \Delta$*?*

Generalized derivations

- $A, B \in \mathbb{M}_n$

Generalized derivations

- $A, B \in \mathbb{M}_n$
- Generalized derivation

$$\delta(T) = BT - TA, \quad T \in \mathbb{M}_n.$$

Generalized derivations

- $A, B \in \mathbb{M}_n$
- Generalized derivation
$$\delta(T) = BT - TA, \quad T \in \mathbb{M}_n.$$
- $\ker\delta = \{T \in \mathbb{M}_n : BT = TA\}$ space of intertwiners.

Generalized derivations

- $A, B \in \mathbb{M}_n$
- Generalized derivation

$$\delta(T) = BT - TA, \quad T \in \mathbb{M}_n.$$

- $\ker \delta = \{T \in \mathbb{M}_n : BT = TA\}$ space of intertwiners.
- $\mathrm{rd}_k(\ker \delta) = 0$ for $k \geq 2$.

Generalized derivations

- $A, B \in \mathbb{M}_n$
- Generalized derivation

$$\delta(T) = BT - TA, \quad T \in \mathbb{M}_n.$$

- $\ker \delta = \{T \in \mathbb{M}_n : BT = TA\}$ space of intertwiners.
- $\mathrm{rd}_k(\ker \delta) = 0$ for $k \geq 2$.
- Zajac:

$$\ker \delta \text{ reflexive} \iff \text{all roots of } \gcd(m_A, m_B) \text{ are simple.}$$

Generalized derivations

- $k \in \mathbb{N}$, J_k the Jordan block of order k,

Generalized derivations

- $k \in \mathbb{N}$, J_k the Jordan block of order k,

- $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$ the Jordan canonical form of A,
- $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$ the Jordan canonical form of B,

Generalized derivations

- $k \in \mathbb{N}$, J_k the Jordan block of order k,
- $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$ the Jordan canonical form of A,
- $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$ the Jordan canonical form of B,
- $R(i, j) := \begin{cases} \frac{1}{2}\min\{r_i, p_j\}\left(\min\{r_i, p_j\} - 1\right) & \text{if } \mu_i = \lambda_j \\ 0 & \text{otherwise.} \end{cases}$

Generalized derivations

- $k \in \mathbb{N}$, J_k the Jordan block of order k,
- $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$ the Jordan canonical form of A,
- $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$ the Jordan canonical form of B,
- $R(i, j) := \begin{cases} \frac{1}{2}\min\{r_i, p_j\}\left(\min\{r_i, p_j\} - 1\right) & \text{if } \mu_i = \lambda_j \\ 0 & \text{otherwise.} \end{cases}$

Proposition

(i) $\mathrm{rd}(\ker \delta) = \sum_{i=1}^{M} \sum_{j=1}^{N} R(i, j)$.

Generalized derivations

- $k \in \mathbb{N}$, J_k the Jordan block of order k,
- $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$ the Jordan canonical form of A,
- $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$ the Jordan canonical form of B,
- $R(i, j) := \begin{cases} \frac{1}{2}\min\{r_i, p_j\}\left(\min\{r_i, p_j\} - 1\right) & \text{if } \mu_i = \lambda_j \\ 0 & \text{otherwise.} \end{cases}$

Proposition

(i) $\mathrm{rd}(\ker \delta) = \sum_{i=1}^{M} \sum_{j=1}^{N} R(i, j)$.

(ii) $\ker \delta$ *reflexive* $\Leftrightarrow$ *all roots of* $\gcd(m_A, m_B)$ *are simple.*

Example: $\Delta(T) = BTA - T$

- $A, B \in \mathbb{M}_n$,
- $\epsilon(T) = BTA - T \quad (T \in \mathbb{M}_n)$,
- $\mathrm{rd}(\ker \epsilon) = ?$

Example: $\Delta(T) = BTA - T$

- $A, B \in \mathbb{M}_n$,
- $\epsilon(T) = BTA - T \quad (T \in \mathbb{M}_n)$,
- $\mathrm{rd}(\ker \epsilon) = ?$

Proposition

(i) $1 \notin \sigma(A)\,\sigma(B) \quad \Rightarrow \quad \ker \epsilon$ *reflexive.*

Example: $\Delta(T) = BTA - T$

- $A, B \in \mathbb{M}_n$,
- $\epsilon(T) = BTA - T \quad (T \in \mathbb{M}_n)$,
- $\mathrm{rd}(\ker \epsilon) = ?$

Proposition

(i) $1 \notin \sigma(A)\sigma(B) \quad \Rightarrow \quad \ker \epsilon$ *reflexive.*

(ii) $\mathrm{rd}(\ker \epsilon) = \sum_{\mu_i \lambda_j = 1} \frac{1}{2} \min\{r_i, p_j\} \left(\min\{r_i, p_j\} - 1\right)$.

Proof.

- A similar to $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$,
- B similar to $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$.

Proof.

- A similar to $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$,
- B similar to $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$.

- Define $\epsilon_{ij}(T) = (\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) - T$,
  - elementary operator on $\mathbb{M}_{r_i,p_j}$, the space of all r_i-by-p_j complex matrices.

Proof.

- A similar to $(\lambda_1 + J_{p_1}) \oplus \ldots \oplus (\lambda_N + J_{p_N})$,
- B similar to $(\mu_1 + J_{r_1}) \oplus \ldots \oplus (\mu_M + J_{r_M})$.
- Define $\epsilon_{ij}(T) = (\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) - T$,
  - elementary operator on $\mathbb{M}_{r_i,p_j}$, the space of all r_i-by-p_j complex matrices.

$$\Downarrow$$

$$\mathrm{rd}(\ker \epsilon) = \sum_{i=1}^{M} \sum_{j=1}^{N} \mathrm{rd}(\ker \epsilon_{ij}).$$

□

Proof (cont.)

- $(\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) = T \qquad (T \in \mathbb{M}_{r_i,p_j}),$
  - T eigenvector of $(\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})$ at eigenvalue 1.

Proof (cont.)

- $(\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) = T \quad (T \in \mathbb{M}_{r_i,p_j})$,
  - T eigenvector of $(\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})$ at eigenvalue 1.
- $\sigma\left((\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})\right) = \{\lambda_j\, \mu_i\}$,

Proof (cont.)

- $(\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) = T \quad (T \in \mathbb{M}_{r_i,p_j})$,
  - T eigenvector of $(\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})$ at eigenvalue 1.
- $\sigma\left((\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})\right) = \{\lambda_j\, \mu_i\}$,
- $\lambda_j \mu_i \neq 1 \quad \Rightarrow \quad \ker \epsilon_{ij} = \{0\}$,

Proof (cont.)

- $(\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) = T \qquad (T \in \mathbb{M}_{r_i,p_j})$,
  - T eigenvector of $(\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})$ at eigenvalue 1.
- $\sigma\left((\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})\right) = \{\lambda_j\, \mu_i\}$,
- $\lambda_j \mu_i \neq 1 \quad \Rightarrow \quad \ker \epsilon_{ij} = \{0\}$,
- $\lambda_j \mu_i = 1 \quad \Rightarrow \quad \mu_i + J_{r_i}$ and $\lambda_j + J_{p_j}$ invertible.

Proof (cont.)

- $(\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) = T \quad (T \in \mathbb{M}_{r_i,p_j})$,
  - T eigenvector of $(\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})$ at eigenvalue 1.
- $\sigma\left((\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})\right) = \{\lambda_j\, \mu_i\}$,
- $\lambda_j \mu_i \neq 1 \quad \Rightarrow \quad \ker \epsilon_{ij} = \{0\}$,
- $\lambda_j \mu_i = 1 \quad \Rightarrow \quad \mu_i + J_{r_i}$ and $\lambda_j + J_{p_j}$ invertible.
  - Define $\tilde{\epsilon}_{ij}\,(T) = (\mu_i + J_{r_i})T - T(\lambda_j + J_{p_j})^{-1}$.

Proof (cont.)

- $(\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) = T \quad (T \in \mathbb{M}_{r_i,p_j})$,
    - T eigenvector of $(\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})$ at eigenvalue 1.
- $\sigma\left((\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})\right) = \{\lambda_j\, \mu_i\}$,
- $\lambda_j\mu_i \neq 1 \quad \Rightarrow \quad \ker \epsilon_{ij} = \{0\}$,
- $\lambda_j\mu_i = 1 \quad \Rightarrow \quad \mu_i + J_{r_i}$ and $\lambda_j + J_{p_j}$ invertible.
    - Define $\tilde{\epsilon}_{ij}\,(T) = (\mu_i + J_{r_i})T - T(\lambda_j + J_{p_j})^{-1}$.
    - $\ker \epsilon_{ij} = \ker \tilde{\epsilon}_{ij}$,

Proof (cont.)

- $(\mu_i + J_{r_i})\, T\, (\lambda_j + J_{p_j}) = T \quad (T \in \mathbb{M}_{r_i,p_j})$,
  - T eigenvector of $(\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})$ at eigenvalue 1.
- $\sigma\left((\lambda_j + J_{p_j})^{\mathsf{T}} \otimes (\mu_i + J_{r_i})\right) = \{\lambda_j\, \mu_i\}$,
- $\lambda_j \mu_i \neq 1 \quad \Rightarrow \quad \ker \epsilon_{ij} = \{0\}$,
- $\lambda_j \mu_i = 1 \quad \Rightarrow \quad \mu_i + J_{r_i}$ and $\lambda_j + J_{p_j}$ invertible.
  - Define $\tilde{\epsilon}_{ij}(T) = (\mu_i + J_{r_i})T - T(\lambda_j + J_{p_j})^{-1}$.
  - $\ker \epsilon_{ij} = \ker \tilde{\epsilon}_{ij}$,
  - $\mathrm{rd}(\ker \epsilon_{ij}) = \frac{1}{2} \min\{r_i, p_j\} (\min\{r_i, p_j\} - 1)$.

□

Image of an elementary operator

- $(A_1, \ldots, A_k)$, $(B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$,

Image of an elementary operator

- $(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$,
- $\Delta(T) = B_1TA_1 + B_2TA_2 + \ldots + B_kTA_k \qquad (T \in L(\mathcal{X})),$

Image of an elementary operator

- $(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$,
- $\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k \qquad (T \in L(\mathcal{X}))$,
- $\ker \Delta$ is k-reflexive, i.e., $\mathrm{rd}_k(\ker \Delta) = 0$.

Image of an elementary operator

- $(A_1, \dots, A_k), (B_1, \dots, B_k)$ k-tuples of operators on $\mathcal{X}$,
- $\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k \qquad (T \in L(\mathcal{X}))$,
- $\ker \Delta$ is k-reflexive, i.e., $\mathrm{rd}_k(\ker \Delta) = 0$.

Question

Is $\mathrm{im}\,\Delta$ *also* k*-reflexive?*

Image of an elementary operator

- $(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$,
- $\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k \qquad (T \in L(\mathcal{X}))$,
- $\ker \Delta$ is k-reflexive, i.e., $\mathrm{rd}_k(\ker \Delta) = 0$.

Question

Is $\mathrm{im}\,\Delta$ also k-reflexive?

Example

- $A, B \in \mathbb{M}_n$,

Image of an elementary operator

- $(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$,
- $\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k \qquad (T \in L(\mathcal{X}))$,
- $\ker \Delta$ is k-reflexive, i.e., $\mathrm{rd}_k(\ker \Delta) = 0$.

Question

Is $\mathrm{im}\,\Delta$ also k-reflexive?

Example

- $A, B \in \mathbb{M}_n$,
- $\tau(T) = BTA \quad (T \in \mathbb{M}_n)$,

Image of an elementary operator

- $(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ k-tuples of operators on $\mathcal{X}$,
- $\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k \qquad (T \in L(\mathcal{X}))$,
- $\ker \Delta$ is k-reflexive, i.e., $\mathrm{rd}_k(\ker \Delta) = 0$.

Question

Is $\mathrm{im}\, \Delta$ also k-reflexive?

Example

- $A, B \in \mathbb{M}_n$,
- $\tau(T) = BTA \quad (T \in \mathbb{M}_n)$,
- $\ker \tau$ and $\mathrm{im}\, \tau$ are reflexive spaces.

Image of an elementary operator

Definition

Annihilator of a nonempty subset $\mathcal{S} \subseteq \mathbb{M}_n$

$$\mathcal{S}_{\perp} = \{C \in \mathbb{M}_n : \operatorname{tr}(CS) = 0 \text{ for all } S \in \mathcal{S}\}.$$

Image of an elementary operator

Definition

Annihilator of a nonempty subset $\mathcal{S} \subseteq \mathbb{M}_n$

$$\mathcal{S}_\perp = \{C \in \mathbb{M}_n : \operatorname{tr}(CS) = 0 \text{ for all } S \in \mathcal{S}\}.$$

Lemma

$\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k$ *elementary operator on* $\mathbb{M}_n$.

Image of an elementary operator

Definition

Annihilator of a nonempty subset $\mathcal{S} \subseteq \mathbb{M}_n$

$$\mathcal{S}_\perp = \{C \in \mathbb{M}_n : \operatorname{tr}(CS) = 0 \text{ for all } S \in \mathcal{S}\}.$$

Lemma

$\Delta(T) = B_1 T A_1 + B_2 T A_2 + \ldots + B_k T A_k$ *elementary operator on* $\mathbb{M}_n$.

$$\Downarrow$$

There exists an elementary operator $\tilde{\Delta}$ *such that* $(\mathrm{im}\,\Delta)_\perp = \ker \tilde{\Delta}$.

Proof.

Define

$$\tilde{\Delta}(T) = A_1 T B_1 + \ldots + A_k T B_k \qquad (T \in \mathbb{M}_n).$$

Proof.

Define

$$\tilde{\Delta}(T) = A_1 T B_1 + \ldots + A_k T B_k \qquad (T \in \mathbb{M}_n).$$

If $T \in \mathbb{M}_n$, then

$$\operatorname{tr}(\Delta(T)C) = \operatorname{tr}(B_1 T A_1 C) + \ldots + \operatorname{tr}(B_k T A_k C)$$

Proof.

Define

$$\tilde{\Delta}(T) = A_1 T B_1 + \ldots + A_k T B_k \qquad (T \in \mathbb{M}_n).$$

If $T \in \mathbb{M}_n$, then

$$\begin{aligned} \mathrm{tr}\,(\Delta(T)C) &= \mathrm{tr}\,(B_1 T A_1 C) + \ldots + \mathrm{tr}\,(B_k T A_k C) \\ &= \mathrm{tr}\,(T A_1 C B_1) + \ldots + \mathrm{tr}\,(T A_k C B_k) \\ &= \mathrm{tr}\,(T(A_1 C B_1 + \ldots + A_k C B_k)) \\ &= \mathrm{tr}\,(T\tilde{\Delta}(C)). \end{aligned}$$

Proof.

Define

$$\tilde{\Delta}(T) = A_1 T B_1 + \ldots + A_k T B_k \qquad (T \in \mathbb{M}_n).$$

If $T \in \mathbb{M}_n$, then

$$\begin{aligned}
\mathrm{tr}\,(\Delta(T)C) &= \mathrm{tr}\,(B_1 T A_1 C) + \ldots + \mathrm{tr}\,(B_k T A_k C) \\
&= \mathrm{tr}\,(T A_1 C B_1) + \ldots + \mathrm{tr}\,(T A_k C B_k) \\
&= \mathrm{tr}\,(T(A_1 C B_1 + \ldots + A_k C B_k)) \\
&= \mathrm{tr}\,(T\tilde{\Delta}(C)).
\end{aligned}$$

Hence

$$C \in (\mathrm{im}\,\Delta)_\perp$$

Proof.

Define

$$\tilde{\Delta}(T) = A_1 T B_1 + \ldots + A_k T B_k \qquad (T \in \mathbb{M}_n).$$

If $T \in \mathbb{M}_n$, then

$$\begin{aligned} \mathrm{tr}\,(\Delta(T)C) &= \mathrm{tr}\,(B_1 T A_1 C) + \ldots + \mathrm{tr}\,(B_k T A_k C) \\ &= \mathrm{tr}\,(T A_1 C B_1) + \ldots + \mathrm{tr}\,(T A_k C B_k) \\ &= \mathrm{tr}\,(T(A_1 C B_1 + \ldots + A_k C B_k)) \\ &= \mathrm{tr}\,(T\tilde{\Delta}(C)). \end{aligned}$$

Hence

$$C \in (\mathrm{im}\,\Delta)_\perp \qquad \Leftrightarrow \qquad \tilde{\Delta}(C) \in (\mathbb{M}_n)_\perp = \{0\}$$

Proof.

Define

$$\tilde{\Delta}(T) = A_1 T B_1 + \ldots + A_k T B_k \qquad (T \in \mathbb{M}_n).$$

If $T \in \mathbb{M}_n$, then

$$\begin{aligned} \operatorname{tr}(\Delta(T)C) &= \operatorname{tr}(B_1 T A_1 C) + \ldots + \operatorname{tr}(B_k T A_k C) \\ &= \operatorname{tr}(T A_1 C B_1) + \ldots + \operatorname{tr}(T A_k C B_k) \\ &= \operatorname{tr}(T(A_1 C B_1 + \ldots + A_k C B_k)) \\ &= \operatorname{tr}(T \tilde{\Delta}(C)). \end{aligned}$$

Hence

$$\begin{aligned} C \in (\operatorname{im} \Delta)_\perp \quad &\Leftrightarrow \quad \tilde{\Delta}(C) \in (\mathbb{M}_n)_\perp = \{0\} \\ &\Leftrightarrow \quad C \in \ker \tilde{\Delta}. \quad \square \end{aligned}$$

Image of an elementary operator

- F_k all elements in $\mathbb{M}_n$ of rank k or less,

Image of an elementary operator

- F_k all elements in $\mathbb{M}_n$ of rank k or less,
- $\mathrm{Ref_k}\mathcal{S} = (\mathcal{S}_\perp \cap F_k)_\perp$.

Image of an elementary operator

- F_k all elements in $\mathbb{M}_n$ of rank k or less,
- $\mathrm{Ref_k}\mathcal{S} = (\mathcal{S}_\perp \cap F_k)_\perp$.

Example

There exists a generalized derivation δ on $\mathbb{M}_3$ such that $\mathrm{im}\,\delta$ is not 2-reflexive.

Image of an elementary operator

- F_k all elements in $\mathbb{M}_n$ of rank k or less,
- $\mathrm{Ref_k}\mathcal{S} = (\mathcal{S}_\perp \cap F_k)_\perp$.

Example

There exists a generalized derivation δ on $\mathbb{M}_3$ such that $\mathrm{im}\,\delta$ is not 2-reflexive.

- $\delta(T) = J_3T - TJ_3 \quad (T \in \mathbb{M}_3) \quad \Rightarrow \quad \mathrm{im}\,\delta$ is 3-reflexive.

Image of an elementary operator

- F_k all elements in $\mathbb{M}_n$ of rank k or less,
- $\mathrm{Ref_k}\mathcal{S} = (\mathcal{S}_\perp \cap F_k)_\perp$.

Example

There exists a generalized derivation δ on $\mathbb{M}_3$ such that $\operatorname{im}\delta$ is not 2-reflexive.

- $\delta(T) = J_3 T - T J_3 \quad (T \in \mathbb{M}_3) \quad \Rightarrow \quad \operatorname{im}\delta$ is 3-reflexive.
- $(\operatorname{im}\delta)_\perp = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$.

Image of an elementary operator

- F_k all elements in $\mathbb{M}_n$ of rank k or less,
- $\mathrm{Ref_k}\mathcal{S} = (\mathcal{S}_\perp \cap F_k)_\perp$.

Example

There exists a generalized derivation δ on $\mathbb{M}_3$ such that $\operatorname{im}\delta$ is not 2-reflexive.

- $\delta(T) = J_3 T - T J_3 \quad (T \in \mathbb{M}_3) \quad \Rightarrow \quad \operatorname{im}\delta$ is 3-reflexive.
- $(\operatorname{im}\delta)_\perp = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$.
- $(\operatorname{im}\delta)_\perp \cap F_2 \subsetneq (\operatorname{im}\delta)_\perp \quad \Rightarrow \quad \operatorname{im}\delta \subsetneq \mathrm{Ref}_2(\operatorname{im}\delta)$.

Image of an elementary operator

- F_k all elements in $\mathbb{M}_n$ of rank k or less,
- $\mathrm{Ref}_\mathrm{k}\mathcal{S} = (\mathcal{S}_\perp \cap F_k)_\perp$.

Example

There exists a generalized derivation δ on $\mathbb{M}_3$ such that $\mathrm{im}\,\delta$ is not 2-reflexive.

- $\delta(T) = J_3 T - T J_3 \quad (T \in \mathbb{M}_3) \quad \Rightarrow \quad \mathrm{im}\,\delta$ is 3-reflexive.
- $(\mathrm{im}\,\delta)_\perp = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$.
- $(\mathrm{im}\,\delta)_\perp \cap F_2 \subsetneq (\mathrm{im}\,\delta)_\perp \quad \Rightarrow \quad \mathrm{im}\,\delta \subsetneq \mathrm{Ref}_2(\mathrm{im}\,\delta)$.
  - $\mathrm{im}\,\delta$ is not 2-reflexive,

Image of an elementary operator

- F_k all elements in $\mathbb{M}_n$ of rank k or less,
- $\mathrm{Ref_k}\mathcal{S} = (\mathcal{S}_\perp \cap F_k)_\perp$.

Example

There exists a generalized derivation δ on $\mathbb{M}_3$ such that $\operatorname{im}\delta$ is not 2-reflexive.

- $\delta(T) = J_3 T - T J_3 \quad (T \in \mathbb{M}_3) \quad \Rightarrow \quad \operatorname{im}\delta$ is 3-reflexive.
- $(\operatorname{im}\delta)_\perp = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$.
- $(\operatorname{im}\delta)_\perp \cap F_2 \subsetneq (\operatorname{im}\delta)_\perp \quad \Rightarrow \quad \operatorname{im}\delta \subsetneq \mathrm{Ref}_2(\operatorname{im}\delta)$.
  - $\operatorname{im}\delta$ is not 2-reflexive,
  - $\mathrm{rd}(\operatorname{im}\delta) = 2, \quad \mathrm{rd}_2(\operatorname{im}\delta) = 1$.

J. Bračič, B. Kuzma: *Reflexivity defect of spaces of linear operators*, Linear Algebra Appl. **430** (2009), 344-359.

M. Zajac: *On reflexivity and hyperreflexivity of some spaces of intertwining operators*, Math. Bohem. **133** (2008), 75-83.

M. Zajac: *Reflexivity of intertwining operators in finite dimensional spaces*, unpublished.

Thank you for your attention.