On operators with $C_{.0}$ nonunitary part

Patryk Pagacz

Department of Mathematics,
Jagiellonian University

Workshop on Functional Analysis and its Applications in Mathematical Physics and Optimal Control
September 5-10, 2011, Nemecka(Slovak Republic)

$C_{.0}$ contractions

Definition

Is a contraction T which adjoints T^* strongly stable $(T^{*n} \longrightarrow 0)$ it is called $C_{.0}$ contraction.

$C_{.0}$ contractions

Definition

Is a contraction T which adjoints T^* strongly stable ($T^{*n} \to 0$) it is called $C_{.0}$ contraction.

Remark

T is $C_{.0}$ contraction if and only if $\|T^{*n}x\| \to 0$ for each $x \in \mathcal{H}$.

Which operators have the $C_{.0}$ nonunitary part?

Criterion

Definition

Let the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *such that* $Tx_{n+1} = x_n$ *be called a backward sequence of* T.

Criterion

Definition

Let the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *such that* $Tx_{n+1} = x_n$ *be called a backward sequence of* T.

Remark

The norms of the backward sequence of any contraction are increasing.

Criterion

Definition

Let the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *such that* $Tx_{n+1} = x_n$ *be called a backward sequence of* T.

Remark

The norms of the backward sequence of any contraction are increasing.

Theorem

Let T *be a contraction. The following conditions are equivalent:*

1. *for any bounded backward sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of* T*, the sequence of norms* $\{\|x_n\|\}_{n\in\mathbb{N}}$ *is constant,*
2. *nonunitary part of* T *is of class* $C_{.0}$.

Criterion

For a contraction T the sequence $\{T^n T^{*n}\}_{n\in\mathbb{N}}$ is strongly convergent. Let $\lim\limits_{n\to\infty} T^n T^{*n} = A$

Criterion

For a contraction T the sequence $\{T^n T^{*n}\}_{n\in\mathbb{N}}$ is strongly convergent. Let $\lim_{n\to\infty} T^n T^{*n} = A$

Proposition

$A^{\frac{1}{2}}(\mathcal{H}) = \{x \in \mathcal{H}|$ there is a bounded backward sequence $\{x_n\}_n$ such that $x = x_0\}$,
Moreover, if $x_0 \in A^{\frac{1}{2}}(\mathcal{H})$ then there exists a backward sequence $\{x_n\}_n$ begins with x_0 such that $\|x_n\| \to \inf\{\|z\| : x_0 = A^{\frac{1}{2}}z\}$

Criterion

For a contraction T the sequence $\{T^n T^{*n}\}_{n\in\mathbb{N}}$ is strongly convergent. Let $\lim_{n\to\infty} T^n T^{*n} = A$

Proposition

$A^{\frac{1}{2}}(\mathcal{H}) = \{x \in \mathcal{H}|$ there is a bounded backward sequence $\{x_n\}_n$ such that $x = x_0\}$,
Moreover, if $x_0 \in A^{\frac{1}{2}}(\mathcal{H})$ then there exists a backward sequence $\{x_n\}_n$ begins with x_0 such that $\|x_n\| \to \inf\{\|z\| : x_0 = A^{\frac{1}{2}}z\}$

Proof.

The proof follows the papers of E. Durszt and Z. Sebestyen. □

Criterion

Outline of the criterion's proof

- By the above proposition if $A^{\frac{1}{2}}(z) = x$ for $z \in \overline{A^{\frac{1}{2}}(\mathcal{H})}$ then $\|z\| = \|x\|$

Criterion

Outline of the criterion's proof

- By the above proposition if $A^{\frac{1}{2}}(z) = x$ for $z \in \overline{A^{\frac{1}{2}}(\mathcal{H})}$ then $\|z\| = \|x\|$
- A is a projection

Criterion

Outline of the criterion's proof

- By the above proposition if $A^{\frac{1}{2}}(z) = x$ for $z \in \overline{A^{\frac{1}{2}}(\mathcal{H})}$ then $\|z\| = \|x\|$
- A is a projection
- By [5] $T = U \oplus S \oplus G$, where G is a $C_{.0}$ cnu contraction, S is a backward unilateral shift and U is unitary.

Criterion

Outline of the criterion's proof

- By the above proposition if $A^{\frac{1}{2}}(z) = x$ for $z \in \overline{A^{\frac{1}{2}}(\mathcal{H})}$ then $\|z\| = \|x\|$
- A is a projection
- By [5] $T = U \oplus S \oplus G$, where G is a $C_{.0}$ cnu contraction, S is a backward unilateral shift and U is unitary.
- For S we can find a backward sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x_n\|$ is not constant. So $T = U \oplus G$.

The problem

Which classes of contractions have the $C_{.0}$ nonunitary part?

The problem

Which classes of contractions have the $C_{.0}$ nonunitary part?

1. Putnam (1975): Hyponormal operators

The problem

Which classes of contractions have the $C_{.0}$ nonunitary part?

1. Putnam (1975): Hyponormal operators
2. Duggal (1994): p-hyponormal and k-paranormal operators

The problem

Which classes of contractions have the $C_{.0}$ nonunitary part?

1. Putnam (1975): Hyponormal operators
2. Duggal (1994): p-hyponormal and k-paranormal operators
3. (p, k)-quasihyponormal and $k*$-paranormal operators

Hyponormal operators

Definition

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be hyponormal iff $T^*T - TT^* \geq 0$

Hyponormal operators

Definition

An operator $T \in \mathcal{B}(\mathcal{H})$ *is said to be hyponormal iff* $T^*T - TT^* \geq 0$

Remark

T *is hyponormal if and only if* $\|T^*x\| \leq \|Tx\|$ *for each* $x \in \mathcal{H}$

Hyponormal operators

Definition

An operator $T \in \mathcal{B}(\mathcal{H})$ *is said to be hyponormal iff* $T^*T - TT^* \geq 0$

Remark

T *is hyponormal if and only if* $\|T^*x\| \leq \|Tx\|$ *for each* $x \in \mathcal{H}$

Example

Let us $S : l^2 \ni (x_1, x_2, x_3, \ldots) \mapsto (0, \lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots) \in l^2$ *be a weight shift.*
S *is hyponormal if and only if* $\{|\lambda_n|\}_{n \in \mathbb{N}}$ *is increasing.*

Classes richer than the class of hyponormal operators

Definition

$T \in \mathcal{B}(\mathcal{H})$ is said to be (p, k)-quasihyponormal if

$$T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$$

for a positive number $0 < p \leq 1$ and a positive integer k.

Classes richer than the class of hyponormal operators

Definition

$T \in \mathcal{B}(\mathcal{H})$ is said to be (p, k)-quasihyponormal if

$$T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$$

for a positive number $0 < p \leq 1$ and a positive integer k.

Remark

The (p, k)-quasihyponormal operator is (p', k')-quasihyponormal, where $0 < p' \leq p \leq 1$ and $k \leq k'$.

Classes richer than the class of hyponormal operators

Definition

$T \in \mathcal{B}(\mathcal{H})$ is said to be (p, k)-quasihyponormal if

$$T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$$

for a positive number $0 < p \leq 1$ and a positive integer k.

Remark

The (p, k)-quasihyponormal operator is (p', k')-quasihyponormal, where $0 < p' \leq p \leq 1$ and $k \leq k'$.

Example

Each nilpotent is (p, k)-quasihyponormal for some k.

Theorem

If T is (p, k)-quasihyponormal operator then it has the $C_{.0}$ nonunitary part.

Theorem

If T is (p, k)-quasihyponormal operator then it has the $C_{.0}$ nonunitary part.

Proof.

By [4] T satisfies the inequality

$$\|T^k x\|^2 \leq \|T^{k+1}x\| \|T^{k-1}x\|$$

for all unit vectors in $\mathcal{H}$.

Theorem

If T is (p, k)-quasihyponormal operator then it has the $C_{.0}$ nonunitary part.

Proof.

By [4] T satisfies the inequality

$$\|T^k x\|^2 \leq \|T^{k+1}x\|\|T^{k-1}x\|$$

for all unit vectors in $\mathcal{H}$.
So, if we choose a bounded backward sequence, then we obtain

$$\|x_n\|^2 = \|T^k x_{n+k}\|^2 = \|x_{n+k}\|^2 \|T^k \frac{x_{n+k}}{\|x_{n+k}\|}\|^2 \leq$$

$$\leq \|x_{n+k}\|^2 \|T^{k+1} \frac{x_{n+k}}{\|x_{n+k}\|}\|\|T^{k-1} \frac{x_{n+k}}{\|x_{n+k}\|}\| =$$

$$= \|T^{k+1}x_{n+k}\|\|T^{k-1}x_{n+k}\| = \|x_{n-1}\|\|x_{n+1}\|$$

Proof.

$$\|x_n\|^2 \leq \|x_{n-1}\|\|x_{n+1}\|$$

Proof.

$$\|x_n\|^2 \leq \|x_{n-1}\|\|x_{n+1}\|$$

So

$$\frac{\|x_n\|}{\|x_{n-1}\|} \leq \frac{\|x_{n+1}\|}{\|x_n\|}$$

Proof.

$$\|x_n\|^2 \leq \|x_{n-1}\| \|x_{n+1}\|$$

So

$$1 \leq \frac{\|x_n\|}{\|x_{n-1}\|} \leq \frac{\|x_{n+1}\|}{\|x_n\|} \to 1$$

since $\{\|x_n\|\}_{n \in \mathbb{N}}$ increases and converges

Proof.

$$\|x_n\|^2 \leq \|x_{n-1}\| \|x_{n+1}\|$$

So

$$1 \leq \frac{\|x_n\|}{\|x_{n-1}\|} \leq \frac{\|x_{n+1}\|}{\|x_n\|} \to 1$$

since $\{\|x_n\|\}_{n\in\mathbb{N}}$ increases and converges

Thus $\|x_n\| = const$

By our criterion the nonunitary part of T is $C_{.0}$ contraction. □

Classes richer than the class of hyponormal operators

Definition

$T \in \mathcal{B}(\mathcal{H})$ is said to be k∗-paranormal if

$$\|T^*x\|^k \leq \|T^k x\| \|x\|^{k-1}$$

for each $x \in \mathcal{H}$.

Classes richer than the class of hyponormal operators

Definition

$T \in \mathcal{B}(\mathcal{H})$ is said to be k∗-paranormal if

$$\|T^*x\|^k \leq \|T^k x\| \|x\|^{k-1}$$

for each $x \in \mathcal{H}$.

Example

Let us $S : l^2 \ni (x_1, x_2, x_3, ...) \mapsto (0, \lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, ...) \in l^2$ be a weight shift.
Then S is k∗-paranormal if and only if $|\lambda_n|^k \leq |\lambda_{n+1}\lambda_{n+2} \dots \lambda_{n+k}|$ for all $n \in \mathbb{N}_+$

Classes richer than the class of hyponormal operators

Theorem

$k*$-paranormal operators have the $C_{.0}$ nonunitary part.

Classes richer than the class of hyponormal operators

Theorem

$k*$-paranormal operators have the $C_{.0}$ nonunitary part.

Proposition

Each $k*$-paranormal operator is a $(k+1)$-paranormal operator.

Classes richer than the class of hyponormal operators

Theorem

$k*$-paranormal operators have the $C_{.0}$ nonunitary part.

Proposition

Each $k*$-paranormal operator is a $(k+1)$-paranormal operator.

Proof.

Let $T \in \mathcal{B}(\mathcal{H})$ be a $k*$-paranormal operator.

Classes richer than the class of hyponormal operators

Theorem

$k*$-paranormal operators have the $C_{.0}$ nonunitary part.

Proposition

Each $k*$-paranormal operator is a $(k+1)$-paranormal operator.

Proof.

Let $T \in \mathcal{B}(\mathcal{H})$ be a $k*$-paranormal operator.

$$\|Tx\|^{2k} = \langle T^*Tx, x\rangle^k \leq \|T^*Tx\|^k \|x\|^k \leq \|T^k(Tx)\| \|Tx\|^{k-1} \|x\|^k$$

Classes richer than the class of hyponormal operators

Theorem

$k*$-paranormal operators have the $C_{.0}$ nonunitary part.

Proposition

Each $k*$-paranormal operator is a $(k+1)$-paranormal operator.

Proof.

Let $T \in \mathcal{B}(\mathcal{H})$ be a $k*$-paranormal operator.

$$\|Tx\|^{2k} = \langle T^*Tx, x\rangle^k \leq \|T^*Tx\|^k \|x\|^k \leq \|T^k(Tx)\| \|Tx\|^{k-1} \|x\|^k$$

Thus $\|Tx\|^{k+1} \leq \|T^k(Tx)\| \|x\|^k$. □

- B.P. Duggal, *On unitary parts of contractions*, Indian J. Pure Appl. Math. 25(1994), 1243-1227.
- B.P. Duggal, *On characterising contractions with C_{10} pure part*, Integral Equations Operator Theory 27(1997), 314-323.
- B.P. Duggal and C.S. Kubrusly, *Paranormal contractions have property PF*, Far East J. Math. 14(2004), 237-249.
- E. Durszt, *Contractions as restricted shifts*, Acta Sci. Math. (Szeged) 48(1985), 129-134.
- C.S. Kubrusly, P.C.M. Vieira and D.O. Pinto, *A Decomposition for a class of contractions*, Advan. Math. Sci. Appl. 2(1993), 335-343.
- C.S. Kubrusly and P.C.M. Vieira, *Strong stability for cohyponormal operators*, J. Operator Theory 31(1994), 123-127.

- P. Pagacz, *On Wold-type decomposition*, http://ssdnm.mimuw.edu.pl/student/35
- C.R. Putnam, *Hyponormal contractions and strong power convergence*, Pacific J. Math. 57(1975), 531-538.
- Z. Sebestyen, *On range of adjoint operators in Hilbert space*, Acta Sci. Math. (Szeged) 46(1983), 295-298.
- K. Tanahashi, A. Uchiyama and M. Cho, *Isolated points of spectrum of (p,k)-quasihyponormal operators*, Linear Algebra Appl. 382(2004), 221-229.

Thank you for your attention!