On the commutant of asymptotically non-vanishing contractions

György Pál Gehér

Bolyai Institute, University of Szeged

Joint work with László Kérchy

Nemecká, September, 2011

- The main open problems in Operator theory are the invariant and the hyperinvariant subspace problem in Hilbert spaces.
- Many (hyper)invariant subspace theorem use some property of the commutant of the operator. Unfortunately generally we do not know much about the commutant.
- There are some special cases, when the commutant was described completely, for example: finite dimensional operators, normal operators, injective unilateral shifts ...
- One of the main methods of examining non-normal operators is the theory of contractions. This area was developed by Béla Szőkefalvi-Nagy and Ciprian Foias from the dilatation theorem of Béla Szőkefalvi-Nagy.

- They classified the contractions according to their asymptotical behaviours.
- They got strong structural results in the case when the contraction and its adjoint are simultaneously asymptotically non-vanishing.
- But basic questions are still open, when we know only that the contraction is asymptotically non-vanishing.
- In this case one can associate a unitary asymptote to the contraction on a natural way. This unitary asymptote tells a lot about the contraction.

- Let $\mathcal{H}$ and $\mathcal{K}$ are **separable** Hilbert spaces,
- $T \in \mathcal{L}(\mathcal{H})$ is a **contraction** ($\|T\| \leq 1$),
- $\mathbb{D} := \{\lambda \in \mathbb{C} \colon |\lambda| < 1\}$,
- the **spectrum** of T: $\sigma(T)$,
- the **point-spectrum** of T: $\sigma_p(T)$,
- **the intertwining set** of $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$:

$$\mathcal{I}(A, B) := \{X \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : XA = BX\},$$

- the **commutant** of T:

$$\{T\}' := \mathcal{I}(T, T) = \{C \in \mathcal{L}(\mathcal{H}) : CT = TC\},$$

- the (closed) subspace $\mathcal{M}$ is **hyperinvariant** for T if it is invariant for all $C \in \{T\}'$.

Stable vectors and subspace

- The vector $h \in \mathcal{H}$ is **stable** for T if

$$T^n h \to 0.$$

- The set of stable vectors: $\mathcal{H}_0 = \mathcal{H}_0(T)$,
- $\mathcal{H}_0$ is a hyperinvariant subspace for T. We call it the **stable subspace**.
- The orthogonal complement of $\mathcal{H}_0$ is denoted by $\mathcal{H}_1$, i. e.: $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.

One can classify the contractions with the stable subspaces of T and T^*.

The classes

- $\mathcal{H}_0 = \mathcal{H}$: T is **stable**, $T \in C_{0\cdot}(\mathcal{H})$,
- $\mathcal{H}_0 \neq \mathcal{H}$: T is **asymptotically non-vanishing**, $T \in C_{*\cdot}(\mathcal{H})$,
- $\mathcal{H}_0 = \{0\}$: T is **asymptotically strongly non-vanishing**, $T \in C_{1\cdot}(\mathcal{H})$.

- If $T^* \in C_{i\cdot}(\mathcal{H})$ $(i \in \{0, *, 1\})$, then we say T is of class $C_{\cdot i}$,
- $C_{ij}(\mathcal{H}) := C_{i\cdot}(\mathcal{H}) \cap C_{\cdot j}(\mathcal{H})$ $(i, j \in \{0, *, 1\})$.
- Szőkefalvi-Nagy and Foias got the structure theorems, mentioned in the introduction, in the case when $T \in C_{**}(\mathcal{H})$.

Canonical triangulation

Theorem (C. Foias and B. Sz-Nagy)

Every contraction $T \in \mathcal{L}(\mathcal{H})$ has the matrix form

$$\begin{bmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{bmatrix},$$

where the matrix is meant in the decomposition $\mathcal{H}_0 \oplus \mathcal{H}_1$, $T_{00} \in C_{0.}(\mathcal{H}_0)$ and $T_{11} \in C_{1.}(\mathcal{H}_1)$.

T_{00} is the **stable component** of T.

The commutant mapping

With the sequence $\{(T^*)^n T^n\}_{n\in\mathbb{N}}$ we can construct a universal unitary intertwining pair: (X, W), where W is unitary. This pair is the **unitary asymptote** of T. The mapping

$$\gamma_T = \gamma\colon \{T\}' \to \{W\}', C \mapsto D, \text{ where } XC = DX,$$

is well-defined, and it is a contractive algebra homomorphism. We call it the **commutant mapping** of T.

Purpose

*Examining the **injectivity** of γ.*

Lemma

γ is injective $\iff$ $C \in \{T\}'$, $\operatorname{ran} C \subset \mathcal{H}_0$ implies $C = 0$.

This gives:

- $T \in C_{1.}(\mathcal{H}) \Longrightarrow \gamma$ is injective,
- $T \in C_{0.}(\mathcal{H}) \Longrightarrow \gamma$ is degenerated: $\gamma \equiv 0$.

Question

*What if $\mathcal{H}_0$ is a **proper** subspace?*
Is there a $T \in C_{*.}(\mathcal{H}) \setminus C_{1.}(\mathcal{H})$, for wich γ is injective?

We will assume, that $\mathcal{H}_0$ is proper.

Reducing to the examination of an operator equation

Theorem

If $T \in C_{*.}(\mathcal{H}) \setminus C_{1.}(\mathcal{H})$, then the followings are equivalent:

1. γ *is injective,*
2. *if the operator equation*

$$T_{00}C_{01} - C_{01}T_{11} = C_{00}T_{01}$$

has a solution with $C_{01} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$ *and* $C_{00} \in \{T_{00}\}'$
$\implies$ *this is trivial i.e.:* $C_{00} = 0$ *and* $C_{01} = 0$.

Quasisimilar contractions

The notion of quasisimilarity is very important in Operator Theory. It plays an important role in the canonical model of many operator classes. (We will see later some of these). This is why the following result is so important.

Theorem

If the contractions $T \in \mathcal{L}(\mathcal{H})$ and $T' \in \mathcal{L}(\mathcal{K})$ are quasisimilar to each other, then $\gamma_{T'}$ is injective $\iff$ γ_T is.

Commutant mapping of an orthogonal sum

Let $\{T_i \in \mathcal{L}(\mathcal{H}_i)\}_{i\in\mathcal{N}}$ be a countable system of contractions, $\gamma_i := \gamma_{T_i}$, $T := \sum_{i\in\mathcal{N}} \oplus T_i \in \mathcal{L}\left(\mathcal{H} = \sum_{i\in\mathcal{N}} \oplus \mathcal{H}_i\right)$ and $\gamma := \gamma_T$.

Theorem

1. *If γ_T is injectiv $\Longrightarrow$ γ_i is injective $\forall\, i$.*
2. *The previous point is generally irreversible. We gave a counterexample, where γ_i is injective $\forall\, i$, but γ is not.*
3. *A special case, when the firs point is reversible: if $T_i = T_j$, for all i, j, then γ_i is injective $\iff$ γ_T is.*

Finite co-dimensional stable subspace

Theorem

If $\dim \mathcal{H}_1 < \infty$ *then* γ *is injective* $\iff T \in C_{1\cdot}(\mathcal{H})$.

A spectral condition

We denote the condition $\sigma_p(T) \cap \overline{\sigma_p(T^*)} \cap \mathbb{D} = \emptyset$ with (S). This will be important for us.

Theorem (Necessary conditions)

If γ_T is injective, then the followings hold:

1. $\mathcal{I}(T_{11}, T_{00}) = \{0\}$,
2. $\sigma_r(T_{00}) \cap \sigma_l(T_{11}) \neq \emptyset$,
3. *(S) is satisfied, and*
4. *there is no such direct decomposition $\mathcal{H} = \mathcal{M}_0 \dot{+} \mathcal{M}_1$, where $\mathcal{M}_0, \mathcal{M}_1$ are invariant subspaces of T and $\{0\} \neq \mathcal{M}_0 \subset \mathcal{H}_0(T)$.*

- A quite complicated example can be provided for a contraction such that these four previous conditions are satisfied, but the commutant mapping is not injective.

- But the condition (S) is equivalent to the injectivity in surprisingly many cases.

Simple stable component

Theorem

If $T_{00} = \lambda$ ($\lambda \in \mathbb{D}$), then γ is injective $\iff$ (S) holds.

- For any $\lambda \in \mathbb{D}$ we gave an example such that $T_{00} = \lambda$, but γ is injective.
- So there exists $T \in C_{*.}(\mathcal{H}) \setminus C_{1.}(\mathcal{H})$, wich has injective γ.

Finite dimensional stable subspace

Theorem

If $\dim \mathcal{H}_0 < \infty$, *then*
γ *is injective* $\iff$ *(S) holds.*

The methods of the proof of this theorem can be modified, to prove some results int the general case i.e.: $\dim \mathcal{H}_0 = \dim \mathcal{H}_1 = \aleph_0$.

The root space system

Consider an operator $A \in \mathcal{L}(\mathcal{K})$ and a number $\lambda \in \mathbb{C}$. The λ-**root space** of A is the subspace

$$\widetilde{\ker}(A - \lambda I) := \left(\bigcup_{j=1}^{\infty} \ker\,(A - \lambda I)^j \right)^-$$

The system

$$\left\{ \widetilde{\ker}(A - \lambda I) \colon \lambda \in \sigma_p(A) \right\}$$

of subspaces is called the **root space system** of A.
The root space system of A is said to be **generating**, if

$$\mathcal{K} = \vee \left\{ \widetilde{\ker}(A - \lambda I) \colon \lambda \in \sigma_p(A) \right\}.$$

Theorem

Suppose that $\sigma_p(T_{00}^*) \subset \overline{\sigma_p(T_{00})}$ and the rootspace system of T_{00}^* is generating.
Then γ is injective $\iff$ (S) holds.

Algebraic operators and the calss $\mathcal{C}_0$

- The operator $A \in \mathcal{L}(\mathcal{K})$ is called **algebraic** if there exists a non-zero complex polynomial p such that $p(A) = 0$.
- The c.n.u. contraction A is said to be of class $\mathcal{C}_0$ if there exists a function $0 \neq u \in H^\infty$ such that $u(A) = 0$ (Sz-Nagy–Foias functional calculus).
- For the class $\mathcal{C}_0$ one can define naturally the **minimal function**. This is always an $m_A \in H^\infty$ inner function.
- These two classes have quasisimilar models!

Theorem

If T_{00} is

1. *algebraic, or*
2. *of class $\mathcal{C}_0$ with a Blaschke product minimal function,*

then γ is injective $\iff$ (S) is satisfied.

Summary

1. Our results give useful contributions to the study of the problem.
2. However, the general case left open.
3. Reference: Gy. P. Gehér and L. Kérchy: *On the commutant of asymptotically non-vanishing contractions*, Periodica Mathematica Hungarica, 2011., to appear,

Thanks for your attention.