On the hyperreflexivity of power partial isometries

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Nemecka

$\mathcal{H}$ – complex separable Hilbert space
$B(\mathcal{H})$ – algebra of bounded linear operators on $\mathcal{H}$

$\mathcal{W} \subset B(\mathcal{H})$ – subalgebra with I
$\operatorname{Lat}\mathcal{W} = \{\mathcal{L} \subset \mathcal{H} : A\mathcal{L} \subset \mathcal{L} \text{ for all } A \in \mathcal{W}\}$
$\operatorname{Alg}\operatorname{Lat}\mathcal{W} = \{B \in B(\mathcal{H}) : \operatorname{Lat}\mathcal{W} \subset \operatorname{Lat}B\}$
$\mathcal{W} \subset \operatorname{Alg}\operatorname{Lat}\mathcal{W} \subset B(\mathcal{H})$
$\mathcal{W}$ is *reflexive* $\overset{\text{df}}{\Longleftrightarrow} \mathcal{W} = \operatorname{Alg}\operatorname{Lat}\mathcal{W}$

$A \in B(\mathcal{H}) \qquad \operatorname{dist}(A, \mathcal{W}) = \inf\{\|A - S\| : S \in \mathcal{W}\}$
$\alpha(A, \mathcal{W}) = \sup\{\|(I - P)AP\| : P \in \operatorname{Lat}\mathcal{W}\}$
usually $\qquad \alpha(A, \mathcal{W}) \leqslant \operatorname{dist}(A, \mathcal{W})$

$\mathcal{W}$ is *hyperreflexive* $\overset{\text{df}}{\Longleftrightarrow}$ there is a constant κ such that
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$\mathcal{W} \subset B(\mathcal{H})$
$\mathcal{W}$ hyperreflexive $\Rightarrow$ $\mathcal{W}$ reflexive

$\dim \mathcal{H} < \infty$
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$T \in B(\mathcal{H})$
T is *reflexive (hyperreflexive)* $\Longleftrightarrow$ $\mathcal{W}(T)$ is reflexive (hyperreflexive)
where
$\mathcal{W}(T)$ – the smallest subalgebra of $B(\mathcal{H})$ with I containing T
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S is a *partial isometry* $\Longleftrightarrow SS^*, S^*S$ are projections

S is a *power partial isometry* $\Longleftrightarrow S^n$ is partial isometry for all n

Example

a unilateral shift: $a_s(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$

a backward shift: $a_s^*(x_0, x_1, \dots) = (x_1, x_2, \dots)$

a truncated shift: $a_k(x_0, x_1, \dots, x_{k-1}) = (0, x_0, x_1, \dots, x_{k-2})$.

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Model for power partial isometry (Halmos, Wallen 1970)

S – power partial isometry

$\mathcal{H} = \mathcal{H}_u(S) \oplus \mathcal{H}_s(S) \oplus \mathcal{H}_c(S) \oplus \mathcal{H}_t(S)$

$\mathcal{H}_u(S), \mathcal{H}_s(S), \mathcal{H}_c(S), \mathcal{H}_t(S)$ reduce S and

$S_u = S|_{\mathcal{H}_u(S)}$ – unitary

$S_s = S|_{\mathcal{H}_s(S)}$ – unilateral shift of arbitrary multiplicity

$S_c = S|_{\mathcal{H}_c(S)}$ – backward shift of arbitrary multiplicity

$S_t = S|_{\mathcal{H}_t(S)}$ – possibly infinite direct sum of truncated shifts.

Theorem (Sarason 1968)

The unilateral shift a_s is reflexive.

a_s^* is reflexive

a_k is not reflexive

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Problem

Characterize hyperreflexivity of power partial isometry.

Theorem (Davidson 1987)

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S – power partial isometry
$P_n = S^{*n}S^n$, $Q_n = S^nS^{*n}$ for all positive integers n.
$P_n,\ Q_n$ decreasing
$\overline{d}_k = \dim \mathcal{R}(P_{k-1}(Q_0 - Q_1))$
$d_k = \dim \mathcal{R}(P_{k-1}(Q_0 - Q_1)) \ominus \mathcal{R}(P_k(Q_0 - Q_1))$ for $k \in \mathbb{N}$
$\overline{d}_\infty = d_\infty = \dim \bigcap_{k\in\mathbb{N}} \mathcal{R}(P_{k-1}(Q_0 - Q_1))$

the number $\overline{d}_k$ says how many forward shifts (truncated or not) of order at least k are in operator S,

the number d_k says how many forward shifts (truncated or not) of order exactly k are in operator S.

Symmetrically
$\overline{d}^*_k = \dim \mathcal{R}(Q_{k-1}(P_0 - P_1))$,
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Theorem

Let $S \in B(\mathcal{H})$ be a completely non–unitary power partial isometry.
If

(i) $d_\infty > 0$ *or*

(ii) $d^*_\infty > 0$ *or*

(iii) *there is* $k_0 \in \mathbb{N}$ *such that* $d_k = 0$ *for* $k > k_0$ *and* $d_{k_0} + d_{k_0 - 1} \geqslant 2$

then S is hyperreflexive.

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Proof:

Case 1 – condition (i) is fulfilled.

$\mathcal{W} := \mathcal{W}(a_s \oplus S_t) \subset B(l^2_+ \oplus H)$,

where

$S_t = \oplus_{i=1}^{m} a_{k_i}, \quad H = \oplus_{i=1}^{m} H_{k_i}$ or

$S_t = \oplus_{i=1}^{\infty} a_{k_i}, \quad H = \oplus_{i=1}^{\infty} H_{k_i}.$

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Hadwin made another (but equivalent) approach to hyperreflexivity.

$\mathcal{S} \subset B(\mathcal{H})$ – a weak*-closed subspace.
$\mathcal{S}$ is hyperreflexive if
there is a number κ such that
$\text{ball}\,(\mathcal{S}^{\perp}) \subset \kappa\,\overline{co}(\mathcal{S}^{\perp} \cap \text{ball}\,F_1)$

where
ball – the unit ball
$\mathcal{S}^{\perp}$ – the preannihilator of S
$\overline{co}$ – the closed convex hull
F_1 – the set of rank-one operators

$P, Q \in B(\mathcal{H})$ – projections
$P \bullet Q : B(\mathcal{H}) \to B(\mathcal{H})$
$P \bullet Q(A) = PAQ$ is an idempotent

Theorem (Hadwin)

$\mathcal{S} \subset B(\mathcal{H})$ – weak-closed subspace,*
P, Q – projections,
$P \bullet Q(\mathcal{S})$ – hyperreflexive
$\mathcal{S} \cap \ker P = \{0\}$.
Then
$P \bullet Q \geqslant_{\mathcal{S},r} I_{\mathcal{H}} \bullet I_{\mathcal{H}}$ for some $r > 0 \Rightarrow \mathcal{S}$ is hyperreflexive.

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$(\tau c)^* = B(\mathcal{H})$

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$P \bullet Q : B(\mathcal{H}) \to B(\mathcal{H})$
$(P \bullet Q)^* : \tau c \to \tau c$
$(P \bullet Q(A), t) = (A, (P \bullet Q)^*(t))$
$(P \bullet Q)^*(F_1) \subset F_1$

$P_1 \bullet Q_1$ *r-dominates* $P_2 \bullet Q_2$ with respect to $\mathcal{S}$ $\quad (r > 0)$
$P_1 \bullet Q_1 \geqslant_{\mathcal{S},r} P_2 \bullet Q_2$
iff
$\mathrm{ball}\,((P_2 \bullet Q_2)^*(\tau c)) \subset r\,\overline{\mathrm{co}}(\mathrm{ball}\,((P_1 \bullet Q_1)^*(\tau c)) \cup (\mathcal{S}^\perp \cap \mathrm{ball}\, F_1))$.

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Theorem (Hadwin)

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P_1, Q_1, P_2, Q_2 – projections
$P_1 \bullet Q_2(\mathcal{S}) = \{0\}$,
$P_2 \bullet Q_1(\mathcal{S}) = \{0\}$,
$\pi : P_1 \bullet Q_1(\mathcal{S}) \to P_2 \bullet Q_2(\mathcal{S})$ weak* – continuous,
$\mathcal{S}$ has property $A_1(1)$.
Then $P_1 \bullet Q_1 \geqslant_{\mathcal{S},7r} P_2 \bullet Q_2$ with any $r > \|\pi\|$.

$\mathcal{S}$ has *property* $A_1(1)$ if
for any weak*-continuous functional ϕ on $\mathcal{S}$ there is $g \in F_1$ with $\|g\|_1 \leq (1+\varepsilon)\|\phi\|$ such that $\phi(S) = tr(Sg)$ for $S \in \mathcal{S}$.

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Theorem (Hadwin)

$\mathcal{S} \subset B(\mathcal{H})$– weak*-closed subspace,
P_1, Q_1, P_2, Q_2 – projections
$P_1 \bullet Q_2(\mathcal{S}) = \{0\}$,
$P_2 \bullet Q_1(\mathcal{S}) = \{0\}$,
$\pi : P_1 \bullet Q_1(\mathcal{S}) \to P_2 \bullet Q_2(\mathcal{S})$ weak* – continuous,
$\mathcal{S}$ has property $A_1(1)$.
Then $P_1 \bullet Q_1 \geqslant_{\mathcal{S},7r} P_2 \bullet Q_2$ with any $r > \|\pi\|$.

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$P_1 \in B(l^2_+ \oplus H)$ – projection onto l^2_+
$P_2 \in B(l^2_+ \oplus H)$ – projection onto H.
Clearly $P_1 \bullet P_1 \geqslant_{\mathcal{W},1} P_1 \bullet P_1$.
Also $P_1 \bullet P_1 \geqslant_{\mathcal{W},1} P_1 \bullet P_2$ and $P_1 \bullet P_1 \geqslant_{\mathcal{W},1} P_2 \bullet P_1$.
We can show that
$P_1 \bullet P_1 \geqslant_{\mathcal{W},7r} P_2 \bullet P_2$ for any $r > 1$.
$I_{l^2_+ \oplus H} \bullet I_{l^2_+ \oplus H} = P_1 \bullet P_1 + P_1 \bullet P_2 + P_2 \bullet P_1 + P_2 \bullet P_2$
thus $P_1 \bullet P_1 \geqslant_{\mathcal{W},28r} I_{l^2_+ \oplus H} \bullet I_{l^2_+ \oplus H}$ with any $r > 1$.
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then $\mathcal{W}(a_s \oplus S_t)$ is hyperreflexive.

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