Recent contributions to operator ergodic theory

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joint works with
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An operator $T \in \mathcal{B}(\mathcal{X})$ is called **power bounded** ($T \in PB(\mathcal{X})$) if

$$\sup_{n \geq 1} ||T^n|| < \infty.$$

T is called **Cesàro bounded** ($T \in CB(\mathcal{X})$) if

$$\sup_{n \geq 0} ||M_n(T)|| < \infty.$$

where $M_n(T) = \frac{1}{n+1} \sum_{j=0}^{n} T^j$.

If $T \in PB(\mathcal{X})$ then $T \in CB(\mathcal{X})$. The converse is not true, in general.

Ex: $I - V$ on $L^1(0,1)$, $Vf(t) = \int_0^t f(s)\mathrm{d}s$ the Volterra operator.

A classical theorem of **Gelfand** asserts that if $\sigma(T) = \{1\}$ while T and T^{-1} are power bounded then $T = I$.

Esterle-Katznelson-Tzafriri theorem asserts that if T is power bounded then

$||T^{n+1} - T^n|| = o(1)$ as $n \to \infty$ if and only if $\sigma(T) \cap \mathbb{T} \subset \{1\}$.

Different generalizations of these results were obtained : G. Allan, C. Batty, L. Burlando, D. Drissi, S. Grabiner, L. Kerchy, Z. Leka, O. Nevanlinna, T. Ransford, H. C. Ronnefarth, Y. Tomilov, M. Zarrabi, J. Zemánek.

Allan *: If* $\frac{\|T^n\|}{n} = o(1)$ *and* $\sigma(T) = \{1\}$*, does it follow that* $||T^{n+1} - T^n|| = o(1)$*, as* $n \to \infty$*?*

The answer is no : **Tomilov-Zemánek**
Let $T = I - V$ on $L^1(0,1)$ and

$$\mathcal{T} = \begin{pmatrix} T & T - I \\ 0 & T \end{pmatrix} \quad \text{on} \quad L^1 \oplus L^1,$$

then $\frac{\|\mathcal{T}^n\|}{n} = o(1)$, $\sigma(\mathcal{T}) = \{1\}$, and $\|\mathcal{T}^{n+1} - \mathcal{T}^n\| \to \infty$ as $n \to \infty$.

If $T \in CB(\mathcal{X})$ and $\sigma(T) \cap \mathbb{T} = \{1\}$, does it follow that $||T^{n+1} - T^n|| = o(1)$, as $n \to \infty$?
The answer is no : **Tomilov-Zemánek**

Let $T(k)$ on $L^2(0,1)$, $(T(k)x)(t) = te^{i\frac{(1-t)^{\frac{1}{k}}}{k}}x(t)$, $k \geq 1$
$\mathcal{X} = \bigoplus_{k=1}^{\infty} X_k$, $X_k = L^2 \oplus L^2$, $\mathcal{T} = \bigoplus_{k=1}^{\infty} \mathcal{T}(k)$.

$\mathcal{T} \in CB(\mathcal{X})$, $\frac{\|\mathcal{T}^n\|}{n} = o(1)$, $\sigma(\mathcal{T}) \cap \mathbb{T} = \{1\}$, and
$\|\mathcal{T}(\mathcal{T} - I)^m\| \to \infty$, as $n \to \infty$, for all $m \geq 1$.

Z. Léka : Let $T = I - V$ on $L^2(0,1)$, $\mathcal{T}$ on $L^2 \oplus L^2$. Then $\sigma(T) = \{1\}$, $\frac{\|\mathcal{T}^n\|}{n} = o(1)$, $\mathcal{T} \in CB(L^2 \oplus L^2)$ and $\|\mathcal{T}^{n+1} - \mathcal{T}^n\| \neq o(1)$, but $\|\mathcal{T}^{n+1}x - \mathcal{T}^n x\| = o(1)$, as $n \to \infty$, for all $x \in L^2 \oplus L^2$.

Suciu-Zemánek : Let $\widehat{\mathcal{T}} \in \mathcal{B}(\mathcal{B}(L^2 \oplus L^2))$, $\widehat{\mathcal{T}}S = \mathcal{T}S$, $S \in \mathcal{B}(L^2 \oplus L^2)$. Then $\sigma(\widehat{\mathcal{T}}) = \{1\}$, $\widehat{\mathcal{T}} \in CB(\mathcal{B}(L^2 \oplus L^2))$, $\frac{\|\widehat{\mathcal{T}}^n\|}{n} = o(1)$, and the sequence $\{\widehat{\mathcal{T}}^{n+1} - \widehat{\mathcal{T}}^n\}$ does not converge strongly to 0, but it is a bounded sequence.

Suciu-Zemánek : If $\mathcal{X}$ is a reflexive Banach space and $T \in CB(\mathcal{X})$ with $\sigma(T) \cap \mathbb{T} = \{1\}$ then $\{T^{n+1} - T^n\}$ strongly converges to 0 if and only if it is bounded.

Suciu-Zemánek : If $T \in CB(\mathcal{X})$ and $\sigma(T) \cap \mathbb{T} = \{1\}$ then $\frac{\|T^n\|}{n} = o(1)$, as $n \to \infty$.

If $\mathcal{X}$ is reflexive then $M_n(T)x \to Px$, as $n \to \infty$, for all $x \in \mathcal{X}$, where $P \in \mathcal{B}(\mathcal{X})$ is the projection with $\mathcal{N}(P) = \overline{\mathcal{R}(I - T)}$ and $\mathcal{R}(P) = \mathcal{N}(I - T)$.

T is **Cesàro ergodic** if the sequence $\{M_n(T)\}$ strongly converges in $\mathcal{B}(\mathcal{X})$.

Recall (see H. C. Ronnefarth (1996), J. C. Strikwerda and B. A. Wade (1991)) that for $n, p \in \mathbb{N}$, **the Cesàro means of order** p of $T \in \mathcal{B}(\mathcal{X})$, denoted $M_n^{(p)}(T)$, $n \in \mathbb{N}$, are defined by :
$M_0^{(p)}(T) = I$, $M_n^{(0)}(T) = T^n$ and if $n, p \geq 1$,

$$\begin{aligned} M_n^{(p)}(T) : &= \frac{p}{(n+1)...(n+p)} \sum_{j=0}^{n} \frac{(j+p-1)!}{j!} M_j^{(p-1)}(T) \\ &= \frac{p}{(n+1)...(n+p)} \sum_{j=0}^{n} \frac{(n-j+p-1)!}{(n-j)!} T^j. \end{aligned}$$

Recall that $T \in \mathcal{B}(\mathcal{X})$ is **Abel ergodic** if the Abel average $A_\alpha(T) = (1-\alpha)\sum_{k=0}^{\infty} \alpha^k T^k$ has limit in the strong topology of $\mathcal{B}(\mathcal{X})$ as $\alpha \to 1^-$.

Hille : Let $T \in \mathcal{B}(\mathcal{X})$. Then $s - \lim_{n\to\infty} M_n^{(p)} = P$ if and only if

(i) $s - \lim_{\alpha\to 1^-}(1-\alpha)\sum_{k=0}^{\infty} \alpha^k T^k = P$;

(ii) $s - \lim_{n\to\infty} \frac{T^n}{n^p} = 0$.

Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is **Kreiss bounded** ($T \in KB(\mathcal{X})$) if

$$\sup_{|\lambda|>1}\{(|\lambda|-1)\|(T-\lambda I)^{-1}\|\} < \infty.$$

$PB(\mathcal{X}) \subset KB(\mathcal{X}) \not\subseteq CB(\mathcal{X})$ and $CB(\mathcal{X}) \not\subseteq KB(\mathcal{X})$.

Strikwerda-Wade : $T \in KB(\mathcal{X})$ if and only if $\|M_n^{(2)}(\lambda T)\| = O(1)$, as $n \to \infty$, for every $|\lambda| = 1$.

Nevanlinna, Lin-Shoikhet-Suciu : If $T \in KB(\mathcal{X})$ and $\sigma(T) \cap \mathbb{T} = \{1\}$ then $\frac{\|T^n\|}{n} = o(1)$, as $n \to \infty$.

Theorem

Let $T \in KB(\mathcal{X})$ and $\mathcal{X}$ reflexive. Then $\{M_n^{(p)}(T)\}$ strongly converges in $\mathcal{X}$ for every $p \geq 2$.

Theorem

Let $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{T} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{X})$ be the operator defined by the matrix

$$\mathcal{T} = \begin{pmatrix} T & T - I \\ 0 & T \end{pmatrix}.$$

The following statement hold :
$\mathcal{T}$ is Kreiss bounded if and only if T is Kreiss bounded and $(|\lambda| - 1)|\lambda - 1|\|(T - \lambda I)^{-2}\| = O(1)$ as $|\lambda| \to 1^{+}$.

Corollary

If $T \in KB(\mathcal{X})$ such that $(n+1)||T^n(T-I)|| = O(1)$ as $n \to \infty$, then $\mathcal{T} \in KB(\mathcal{X})$ (of previous Theorem).

Corollary

If $T \in \mathcal{B}(\mathcal{X})$ satisfies $\sqrt{n+1}||T^n(T-I)|| = O(1)$ as $n \to \infty$ and $||M_n^{(2)}(\mathcal{T})|| = O(1)$ as $n \to \infty$ then T is power bounded.

Corollary

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T is called **uniformly Kreiss bounded** if $\|M_n(\lambda T)\| = O(1)$, as $n \to \infty$, for every $|\lambda| = 1$.

Theorem

Let $T \in \mathcal{B}(\mathcal{X})$ and $\mathcal{T} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{X})$ be as in previous Lemma. Then any two of the following statements

(a) T is uniformly Kreiss bounded,

(b) $\|M_n^{(3)}(\lambda T)\| = O(n^{-1})$ as $n \to \infty$, uniformly for λ with $|\lambda| = 1$,

(c) $\mathcal{T}$ is Kreiss bounded,

imply the other statement.

Theorem

Let T be a Cesàro bounded operator on $\mathcal{X}$ which satisfies $\lim_{n\to\infty} \frac{\|T^n x\|}{n} = 0$ for every $x \in \mathcal{X}$. Then T is Cesàro ergodic if and only if

$$(I - T)\overline{(I - T)\mathcal{X}} = (I - T)\mathcal{X}. \tag{1}$$

Theorem

Let $T \in \mathcal{B}(\mathcal{X})$ with $\{\frac{T^n}{n}\}$ bounded satisfy (1) and assume that for some $m \geq 1$ the sequence $\{\frac{1}{n}T^n(T-I)^m\}_{n\geq 1}$ converges to 0 strongly. Then $\lim_{n\to\infty} \frac{\|T^n x\|}{n} = 0$ for any $x \in \mathcal{X}$.

Corollary

Let $T \in \mathcal{B}(\mathcal{X})$ be Cesàro bounded. Then T is Cesàro ergodic if (and only if) T satisfies (1) and for some $m \geq 1$ the sequence $\{\frac{1}{n}T^n(T-I)^m\}_{n\geq 1}$ converges to 0 strongly.

Lemma

Let $T \in \mathcal{B}(\mathcal{X})$. Then $\{M_n(T)(I-T)\}$ is bounded if and only if $\{\frac{1}{n}T^{2n}\}$ is bounded.
Thus, when $1 \notin \sigma(T)$, T is Cesàro bounded if and only if $\{\frac{1}{n}T^{2n}\}$ is bounded.

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Corollary

Let $T \in \mathcal{B}(\mathcal{X})$ be Cesàro bounded. Then T is Cesàro ergodic if (and only if) T satisfies (1) and for some $m \geq 1$ the sequence $\{\frac{1}{n}T^n(T-I)^m\}_{n\geq 1}$ converges to 0 strongly.

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Corollary

Let $T \in \mathcal{B}(\mathcal{X})$ be Cesàro bounded. Then T is Cesàro ergodic if (and only if) T satisfies (1) and for some $m \geq 1$ the sequence $\{\frac{1}{n}T^n(T-I)^m\}_{n\geq 1}$ converges to 0 strongly.

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Let $T \in \mathcal{B}(\mathcal{X})$. Then $\{M_n(T)(I-T)\}$ is bounded if and only if $\{\frac{1}{n}T^{2n}\}$ is bounded.
Thus, when $1 \notin \sigma(T)$, T is Cesàro bounded if and only if $\{\frac{1}{n}T^{2n}\}$ is bounded.

Theorem

Let $T \in \mathcal{B}(\mathcal{X})$ have $r(T) \leq 1$. Then T is uniformly Abel ergodic if and only if it is Abel bounded and $(I - T)\mathcal{X}$ is closed.

Corollary

Let $T \in \mathcal{B}(\mathcal{X})$ be Cesàro bounded. If $(I - T)\mathcal{X}$ is closed, then T is uniformly Abel ergodic.

There exists a Cesàro bounded uniformly Abel ergodic operator which is not uniformly ergodic.

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Theorem

The following are equivalent for $T \in \mathcal{B}(\mathcal{X})$ *:*
(i) $\sup_n \|T^n\|/n < \infty$ *and* T *is uniformly Abel ergodic;*
(ii) T *is Cesàro bounded and* $(I - T)\mathcal{X}$ *is closed.*

Theorem

Let $T \in \mathcal{B}(\mathcal{X})$ *satisfy* $\|T^n\|/n \to 0$. *Then the following conditions are equivalent:*
(i) T *is uniformly ergodic.*
(ii) All the Abel averages A_α, $0 < \alpha < 1$, *are uniformly power convergent to a projection* P *of* $\mathcal{X}$ *onto* $\mathcal{N}(I - T)$, *i.e.*

$$\lim_{n\to\infty} \|A_\alpha^n - P\| \to 0 \text{ for each } \alpha \in (0,1). \tag{2}$$

(iii) For some $\alpha \in (0,1)$ *the operator* A_α *is uniformly power convergent.*

For $T \in \mathcal{B}(\mathcal{X})$ we put

$$S_n(T) = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{k} T^j, \quad n \in \mathbb{N}, \tag{3}$$

and

$$\mathcal{S}(T) = \{x \in \mathcal{X} : \sup_{n \in \mathbb{N}} \|S_n(T)x\| < \infty\}. \tag{4}$$

Theorem

Let $T \in \mathcal{B}(\mathcal{X})$ be Cesàro bounded such that $\{\frac{T^n}{n}\}$ strongly converges to 0. Then $\mathcal{S}(T)$ is closed if and only if $(I - T)\mathcal{X}$ is closed, and in this case T is Cesàro ergodic.

Corollary

If $T \in \mathcal{B}(\mathcal{X})$ is Cesàro ergodic such that $\mathcal{S}(T)$ is closed, then

$$\mathcal{S}(T) = (I - T)\mathcal{X} = \{x \in \mathcal{X} : \{S_n(T)x\} \text{ converges in } \mathcal{X}\}.$$

Theorem

Let $T \in \mathcal{B}(\mathcal{X})$ be Cesàro bounded such that $\{\frac{T^n}{n}\}$ strongly converges to 0. Then $\mathcal{S}(T)$ is closed if and only if $(I - T)\mathcal{X}$ is closed, and in this case T is Cesàro ergodic.

Corollary

If $T \in \mathcal{B}(\mathcal{X})$ is Cesàro ergodic such that $\mathcal{S}(T)$ is closed, then

$$\mathcal{S}(T) = (I - T)\mathcal{X} = \{x \in \mathcal{X} : \{S_n(T)x\} \text{ converges in } \mathcal{X}\}.$$

Theorem

The following are equivalent for a Banach space $\mathcal{X}$:
(i) $\mathcal{X}$ is reflexive;
(ii) Every Cesàro bounded operator T defined on a closed subspace $\mathcal{Y} \subset \mathcal{X}$ such that $\{\frac{T^n}{n}\}$ strongly converges to 0 satisfies

$$(I - T)\mathcal{Y} = \{y \in \mathcal{Y} : \sup_{n \in \mathbb{N}} \|S_n(T)y\| < \infty\}; \tag{5}$$

(iii) Every Cesàro ergodic operator T defined on a closed subspace $\mathcal{Y} \subset \mathcal{X}$ satisfies (5).

Corollary

Let $\mathcal{X}$ be a reflexive Banach space and T be a Cesàro ergodic operator on $\mathcal{X}$. Then T is power-bounded if and only if $\mathcal{S}_0(T) = \mathcal{S}(T)$, if and only if $\mathcal{S}_0(T) = (I - T)\mathcal{X}$, where $\mathcal{S}_0(T) := \{y \in \mathcal{Y} : \sup_{n\in\mathbb{N}} \| \sum_{j=0}^{n} T^j y\|\}$.

Corollary

Let T be a Cesàro bounded operator on $\mathcal{X}$ with $\{\frac{T^n}{n}\}$ strongly convergent to 0. If $\overline{(I - T)\mathcal{X}}$ is a reflexive Banach space then T is Cesàro ergodic.

Corollary

Let $\mathcal{X}$ be a reflexive Banach space and T be a Cesàro ergodic operator on $\mathcal{X}$. Then T is power-bounded if and only if $\mathcal{S}_0(T) = \mathcal{S}(T)$, if and only if $\mathcal{S}_0(T) = (I - T)\mathcal{X}$, where $\mathcal{S}_0(T) := \{y \in \mathcal{Y} : \sup_{n\in\mathbb{N}} \| \sum_{j=0}^{n} T^j y\|\}$.

Corollary

Let T be a Cesàro bounded operator on $\mathcal{X}$ with $\{\frac{T^n}{n}\}$ strongly convergent to 0. If $\overline{(I - T)\mathcal{X}}$ is a reflexive Banach space then T is Cesàro ergodic.

Theorem

For a Banach space $\mathcal{X}$ with a basis the following are equivalent:
(i) *$\mathcal{X}$ is reflexive;*
(ii) *Every Cesàro ergodic operator T on $\mathcal{X}$ satisfies*

$$\mathcal{S}(T) = (I - T)\overline{(I - T)\mathcal{X}};$$

(iii) *Every Cesàro ergodic operator T on $\mathcal{X}$ satisfies* $\mathcal{S}(T) = (I - T)\mathcal{X}$.