Workshop on Functional Analysis and its Applications in Mathematical Physics and Optimal Control

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METHOD OF RELIABLE SOLUTION IN HOMOGENIZATION

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joint work with

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Uncertain data problem and Reliable solution

Mathematical modeling of an engineering problem

- Differential equation(s)
- Boundary and/or initial conditions
- Data of the problem:
  domain and its boundary, coefficients, functions in the equation and in the conditions.

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Problem:

data are not known exactly:

every coefficient can be anywhere within an interval

also geometry is not know exactly

......

Solutions

Stochastic approach

- data: random variables, distribution function, ...
- stochastic differential equations
- complicated theory, ...

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Babuška's idea: Deterministic approach

- full deterministic model
- all possible data are considered
- the worst situation is looked for
- using optimization algorithms

Basic idea

Problem with uncertain data

Reliable solution

Worst scenario method

Basic idea

Problem with uncertain data

Reliable solution

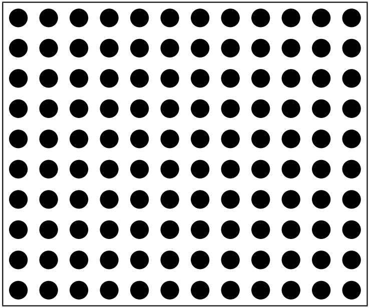
Worst scenario method

- choose a set $\mathscr{U}^{\mathrm{ad}}$ of all admissible data a
- find solution u_a of the problem $(\mathbf{P}[a])$ with data a
- chose a critical functional $\Phi(u)$ on the solution u
- look for the maximum value of $\Phi(u_a)$ for $u \in \mathscr{U}^{\mathrm{ad}}$
- find a the giving maximum value.

I. Hlaváček, J. Chleboun, I. Babuška:
Uncertain input data problems and the worst scenario method, Applied Mathematics and Mechanics, North Holland 2004.

Homogenization

- Physical setting



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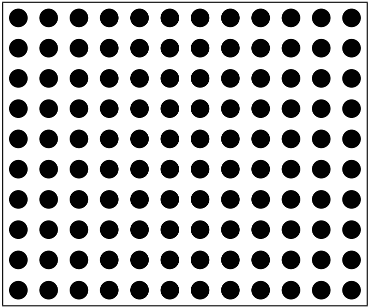

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$$-\mathrm{div}\,(a_p(x)\nabla u_p) = f \qquad\qquad -\,\mathrm{div}\,(b\,\nabla u) = f$$

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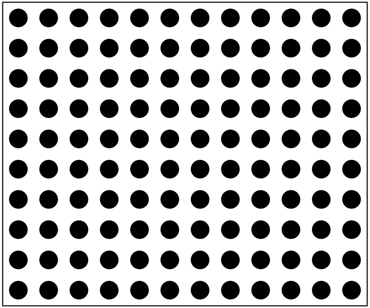


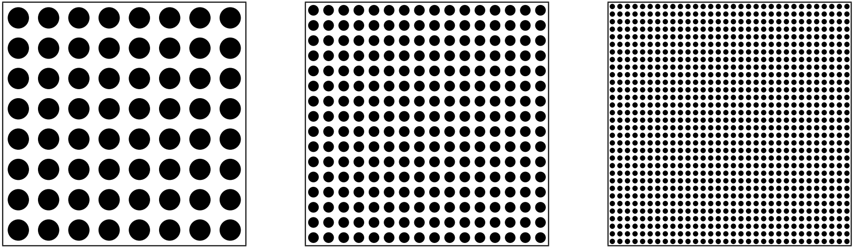
- Mathematical setting

$$-\mathrm{div}\,(a_p(x)\nabla u_p) = f \qquad\qquad -\,\mathrm{div}\,(b\,\nabla u) = f$$

- Computation reason: fine structure needs fine discretization and large number of unknowns and equations.

Homogenization-Mathematical Approach

- Sequence of problems with diminishing period (Babuška 1972)



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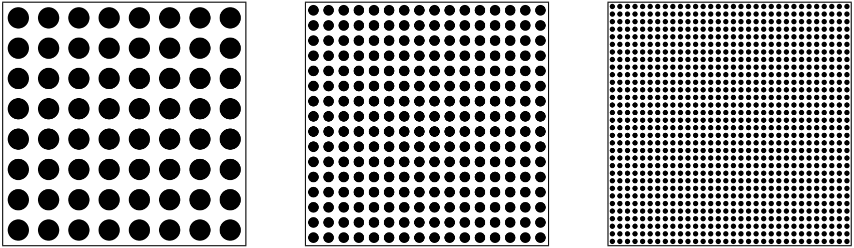

- In the mathematical setting: $\{\varepsilon_h\}, \quad \varepsilon_h \to 0$

$$-\mathrm{div}\,(a^{\varepsilon}(x)u^{\varepsilon}) = f \qquad a^{\varepsilon}(x) = a\left(\frac{x}{\varepsilon}\right) \qquad a(y) - Y\text{−periodic}$$

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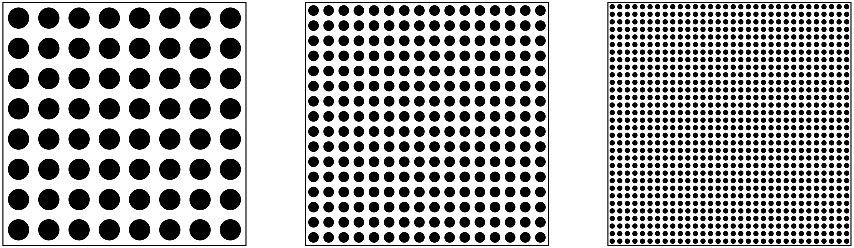

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- Questions:
  - Convergence of the solutions $u^\varepsilon \to u^*$
  - Form of the limit problem $-\mathrm{div}\,(b\,u^*) = f$
  - Formulae for the so-called homogenized coefficients b,

Model problem

Linear elliptic problem

$$-\mathrm{div}\,(a\,\nabla u_a) \equiv -\sum_{i=1,j}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\right) = f \qquad \text{in} \quad \Omega$$

$$u_a = 0 \qquad \text{on} \quad \partial\Omega.$$

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The solution is taken in the so-called weak sense:

PROBLEM $(\mathbf{P}[a])$ *Find a function* $u_a \in W_0^{1,2}(\Omega)$ *satisfying*

$$\boldsymbol{a}_a(u_a, v) \equiv \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u_a}{\partial x_j}\,\frac{\partial v}{\partial x_i}\mathrm{d}x = \int_\Omega f v \mathrm{d}x. \quad \forall\, v \in W_0^{1,2}(\Omega).$$

Assumptions

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Following the Lax-Milgram lemma Problem ($\mathbf{P}[a]$) for $a \in \mathscr{E}(\alpha, M)$ admits unique solution u_a and, in addition,

$$\|u_a\|_{1,2} \leq \frac{1}{\alpha} \|f\|_2 \ .$$

Homogenization – preliminaries

Scale – a sequence $E = \{\varepsilon_n\}_{n=1}^{\infty}$ $\varepsilon_n >$ $\varepsilon \to 0$
The sequences are denoted with a superscript $\varepsilon_n \in E$, $a^{\varepsilon_n} \to a^{\varepsilon}$.

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Basic cell – $Y = \langle 0, 1)^N$.
shifted cells $Y_k = Y + k = \{y + k \mid y \in Y\}$ $k_i \in \mathbb{Z}$ – a pavement of the space $\mathbb{R}^N$,

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Let a be a Y-periodic function, then

$$a^{\varepsilon}(x) = a\left(\frac{x}{\varepsilon}\right) \equiv a\left(\frac{x_1}{\varepsilon}, \ldots, \frac{x_N}{\varepsilon}\right), \quad x \in \Omega$$

is a sequence $\{a^{\varepsilon} \mid \varepsilon \in E\}$ of Y^{ε}-periodic functions on Ω with diminishing period ε.

Homogenization – formulation of the problem

For $\varepsilon \in E$ and a Y-periodic matrix function $a : \Omega \to \mathbb{R}^{N\times N}$ we obtain a ε-periodic functions a_{ij}^{ε} and problem with ε-periodic coefficients:

Problem ($\mathbf{P}[a^{\varepsilon}]$) *Find a function* $u_{a^{\varepsilon}} \in W_0^{1,2}(\Omega)$ *satisfying*

$$\boldsymbol{a}_{a^{\varepsilon}}(u_{a^{\varepsilon}}, v) \equiv \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_{a^{\varepsilon}}}{\partial x_j} \frac{\partial v}{\partial x_i} \mathrm{d}x = \int_{\Omega} f\, v \mathrm{d}x \quad \forall\, v \in W_0^{1,2}(\Omega).$$

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PROBLEM $(\mathbf{P}[a^{\varepsilon}])$ *Find a function* $u_{a^{\varepsilon}} \in W^{1,2}_0(\Omega)$ *satisfying*

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The problem $(\mathbf{P}[a^{\varepsilon}])$ admits unique solution $u_{a^{\varepsilon}}$.

Homogenization – results

Taking a scale $E = \{\varepsilon\}$ we obtain a sequence $\{u_{a^\varepsilon}\}$. The sequence is bounded in $W^{1,2}(\Omega)$.

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- The well known result:

$$u_{a^\varepsilon} \to u_{b^a} \quad \text{weakly in} \quad W^{1,2}(\Omega)$$

- u_{b^a} is a solution to the same type problem but with the so-called homogenized coefficients – matrix of constant function b^a:

Problem ($\mathbf{P}[b^a]$) *Find a function* $u_{b^a} \in W_0^{1,2}(\Omega)$ *satisfying*

$$a_{b^a}(u_{b^a}, v) \equiv \int_\Omega \sum_{i,j=1}^N b_{ij}^a \frac{\partial u_{b^a}}{\partial x_j} \frac{\partial v}{\partial x_i} \mathrm{d}x = \int_\Omega f v \mathrm{d}x. \quad \forall\, v \in W_0^{1,2}(\Omega).$$

Homogenized coefficients

- The homogenized coefficients b^a are given by

$$b^a_{ij} = \int_Y \left[a_{ij}(y) + \sum_{k=1}^{N} a_{ik}(y) \frac{\partial w^k_a}{\partial y_j}(y) \right] \mathrm{d}y,$$

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where w^k_a are Y-periodic solutions to

Problem $(\mathbf{P}_{\mathbf{per}}[a])$ *Find* $w_a = (w^1_a, \ldots, w^N_a)$, $w^k_a \in W^{1,2}_{per}(Y)$:

$$\int_Y \left[\sum_{i,j=1}^{N} a_{ij}(y) \frac{\partial w^k_a}{\partial y_j} \frac{\partial \varphi}{\partial y_i} + \sum_{i=1}^{N} a_{ik}(y) \frac{\partial \varphi}{\partial y_i} \right] \mathrm{d}y = 0 \quad \forall \varphi \in W^{1,2}_{\mathrm{per}}(Y)$$

$\int_Y w^k_a(y)\mathrm{d}y = 0.$

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  $\int_Y w^k_a(y) \mathrm{d}y = 0.$

- The homogenized coefficients b^a_{ij} form also a positive definitive matrix.
- If a_{ij} are symmetric, then the matrix b^a is in the same class $\mathcal{E}(\alpha, M)$.

Uncertain data

- Two component composite material is considered:
  $Y = Y_1 \cup Y_0$ – reinforcing fibres and matrix.
- $$a_{ij}(y) = \begin{cases} p_{ij}^1 & \text{for } y \in Y_1, \\ p_{ij}^0 & \text{for } y \in Y_0 \end{cases}$$
- The set of all such functions $a_{ij}(y)$ with
  $p_{ij}^1 \in I_{ij}^1$ and $p_{ij}^0 \in I_{ij}^0$
  assumed, that it is a subset of $\mathscr{E}(\alpha, M)$
  will be the set of admissible functions $\mathscr{U}^{\text{ad}}$.
- By its construction it is a bounded closed subset in $L^\infty_{\text{per}}(Y)$
- $\mathscr{U}^{\text{ad}}$ is finite dimensional – it is compact

Criterion functional

How to choose the functional Φ evaluating dangerous situations?

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- Another possibility is to test gradient of the homogenized solution u_{b^a}.

Main result

THEOREM. *The functional* Φ *on* $\mathscr{U}^{ad}$ *attains its maximum.*

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Idea of the proof.

- Take a maximizing sequence a_n.
- Due to compactness of $\mathscr{U}^{ad}$ there is a subsequence $a_{n'}$ converging to a^*
- Due to continuity based on estimates $\lim_{n' \to \infty} \Phi(a_{n'}) = \Phi(a^*)$
- a^* yields the maximum value on $\mathscr{U}^{ad}$

Estimates

$$\left|\Phi(a) - \Phi(a')\right| \leq \text{const.} \, \|u_{b^a} - u_{b^{a'}}\|_{W^{1,2}(\Omega)},$$

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$$\max_{i,j} \left| b^a_{ij} - b^{a'}_{ij} \right| \leq \mathrm{const.} \, \|w_a - w_{a'}\|_{W^{1,2}_{\mathrm{per}}(Y,\mathbb{R}^N)},]$$

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$$\|w_a - w_{a'}\|_{W^{1,2}_{\mathrm{per}}(Y)} \leq \text{const.} \, \|a - a'\|_{L^\infty(Y,\mathbb{R}^{N\times N})}.$$

Generalizations

- Problems with strongly monotone operator
- Evolution problems
- uncertainty in geometry
- . . . . . .