Criteria for normality via C_0-semigroups and moment sequences

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(based on a joint paper with I.B. Jung and J. Stochel)

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Introduction

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This leads to a Friedland's result (1982).

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but the implication "$\Rightarrow$" is easy basing on the spectral theorem.

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Then $\{S(t)^*\}_{t \geqslant 0}$ is the proper replacement for e^{tA^*}.

Definitions

A family $\{S(t)\}_{t\geqslant 0}$ of bounded operators is called a C_0**-semigroup** if

- $S(t_1 + t_2) = S(t_1)S(t_2)$,
- $S(0)$ is the identity operator,
- $\lim_{t\to 0+} S(t)f = f$ for all f.

A is called the **infinitesimal generator** of $\{S(t)\}_{t\geqslant 0}$ if

$$Af = \lim_{t\to 0+} \frac{1}{t}(S(t)f - f)$$

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Moreover, if (i) *holds, then* $\mathcal{N}(S(t)) = \{0\}$, $t \geqslant 0$.

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Proof

The proof of (i)⇒(ii). Take $h \in \mathcal{H}$, $\alpha \in (0,1)$, $t_1, t_2 \in [0,\infty)$. Then

$$\|S(\alpha t_1 + (1-\alpha)t_2)h\|^2 = \int_{\mathbb{C}} |\mathrm{e}^{\alpha t_1 \lambda}|^2 \cdot |\mathrm{e}^{(1-\alpha)t_2\lambda}|^2 \, \mu_h(\mathrm{d}\lambda)$$

$$\leqslant \Big(\int_{\mathbb{C}} (|\mathrm{e}^{\alpha t_1 \lambda}|^2)^{\frac{1}{\alpha}} \, \mu_h(\mathrm{d}\lambda) \Big)^{\alpha} \Big(\int_{\mathbb{C}} (|\mathrm{e}^{(1-\alpha)t_2\lambda}|^2)^{\frac{1}{1-\alpha}} \, \mu_h(\mathrm{d}\lambda) \Big)^{1-\alpha}$$

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Moreover, if (i) *holds, then $\mathcal{N}(S(t)) = \{0\}$, $t \geqslant 0$.*

Proof

Proof of (iii)⇒(ii) is based on a lemma stating that a differentiable function on $[a, b)$, which is convex on the left from every point, is convex of class C^1.

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(ii)⇒(i). By convexity we have

$$\log \|S(t)h\| \leqslant \frac{1}{2}\big(\log \|S(0)h\| + \log \|S(2t)h\|\big), \quad h \in \mathcal{H},$$

which means that

$$\|S(t)h\|^2 \leqslant \|h\| \cdot \|S(t)^2 h\|, \quad h \in \mathcal{H},$$

thus $S(t)$ is paranormal. So is $S(t)^*$. Moreover, kernels $S(t)$ and $S(t)^*$ are equal (since both equal to $\{0\}$). By the Ando theorem we get the normality of $S(t)$, hence normality of A.

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Compact case

Proposition

The infinitesimal generator of a compact C_0-semigroup $\{S(t)\}_{t\in[0,\infty)}$ is normal if and only if the function $t \mapsto \log \|S(t)h\|$ is convex on $[0, \infty)$ for every $h \in \mathcal{H}$.

Proof. Convexity $\Rightarrow$ paranormality of $S(t)$ $\Rightarrow$ normality of $S(t)$ (by Istrăţescu, Saitô & Yoshino, 1966) $\Rightarrow$ normality of A.

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Normality via moment sequences

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A bounded operator A is normal if and only if $\mathcal{N}(A) = \mathcal{N}(A^*)$ and for some integers $j, k \geqslant 1$ (equivalently: for all integers $j, k \geqslant 1$) the sequences $\{\|A^n h\|^{2j}\}_{n=0}^{\infty}$ and $\{\|A^{*n} h\|^{2k}\}_{n=0}^{\infty}$ are Hamburger moment sequences for every $h \in \mathcal{H}$.

Proof.

($\Rightarrow$) Mainly by the spectral theorem.

($\Leftarrow$)

$$(\|Ah\|^{2j})^2 = \Big(\int_{\mathbb{R}} t \mu_h(\mathrm{d}t) \Big)^2 \leqslant \int_{\mathbb{R}} t^2 \mu_h(\mathrm{d}t) \int_{\mathbb{R}} t^0 \mu_h(\mathrm{d}t) = \|A^2 h\|^{2j} \|h\|^{2j},$$

so A is paranormal. The same for A^*. By the Ando theorem A is normal. □

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A compact operator A is normal if and only if for some integer $j \geqslant 1$ (equivalently: for all integers $j \geqslant 1$) the sequence $\{\|A^n h\|^{2j}\}_{n=0}^{\infty}$ is a Hamburger moment sequence for every $h \in \mathcal{H}$.

Open question

Fix integer $j \geqslant 2$ and assume that $\{\|A^n h\|^{2j}\}_{n=0}^{\infty}$ is a Hamburger moment sequence for every $h \in \mathcal{H}$. Does it follow that A is subnormal?

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And Now for Something Completely Different

$*54{\cdot}43.\quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51{\cdot}231] \qquad \equiv . \iota\text{‘}x \cap \iota\text{‘}y = \Lambda .$

$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$

$\qquad \vdash :. (\exists x, y) . \alpha = \iota\text{‘}x . \beta = \iota\text{‘}y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.