Optimal Control for a Class of History-dependent Hemivariational Inequalities

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Outline

- Motivation, physical background
- Hyperbolic hemivariational inequality
  - Detailed problem: dynamic viscoelastic contact
  - Subdifferential boundary conditions in mechanics
  - Variational formulation and existence results
- Boltza optimal control problem
- Asymptotic behavior of optimal solutions

Problem setting

Let V and Z be separable Banach spaces and let H be a separable Hilbert space. Suppose

$$V \subset Z \subset H \subset Z^* \subset V^*$$

with dense and continuous embeddings and that $V \subset Z$ compactly.
Let $\gamma \in \mathcal{L}(Z, X)$, where X is a separable Banach space.
Define the spaces
$\mathcal{V} = L^2(0,T;V)$, $\mathcal{Z} = L^2(0,T;Z)$, $\mathcal{W} = \{v \in \mathcal{V} \,:\, v' \in \mathcal{V}^*\}$.

Consider the following second order evolution inclusion:

find $y \in \mathcal{V}$ with $y' \in \mathcal{W}$ such that

$$\begin{cases} y''(t) + A(t, y'(t)) + By(t) + \mathcal{S}y(t) + \\ \qquad\qquad + \gamma^* \partial J(t, \gamma y'(t)) \ni f(t) + E(t)u(t) \;\; \text{a.e. } t \in (0,T) \\ \\ y(0) = y_0, \;\; y'(0) = y_1 \end{cases}$$

where y is a state (displacement) and u is a control.

History-dependent operators

$\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ and there exists $L_S > 0$ such that

$$\|\mathcal{S}y_1(t) - \mathcal{S}y_2(t)\|_{V^*} \le L_S \int_0^t \|y_1(s) - y_2(s)\|_V \, ds$$

$$\forall\, y_1,\, y_2 \in \mathcal{V},\ \text{a.e. } t \in (0,T).$$

Examples

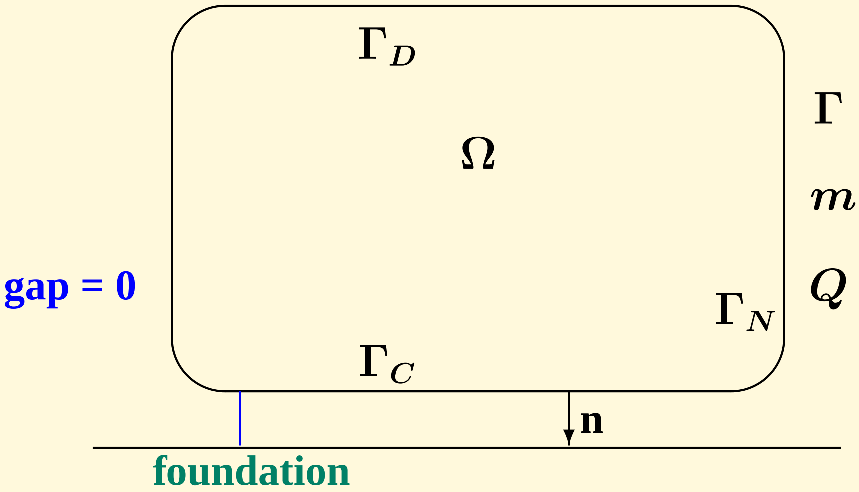
$$\mathcal{S}v(t) = R\Big(\int_0^t v(s)\,ds + v_0\Big) \quad \forall\, v \in \mathcal{V},\ \forall\, t \in (0,T),$$

where $R : V \to V^*$ is a Lipschitz continuous operator and $v_0 \in V$.

$$\mathcal{S}v(t) = \int_0^t C(t-s)\, v(s)\, ds \quad \forall\, v \in \mathcal{V},\ \forall\, t \in (0,T),$$

where $C \in L^2(0,T;\mathcal{L}(V,V^*))$.

Physical setting



$\Gamma = \partial\Omega$

$m(\Gamma_D) > 0$

$Q = \Omega \times (0, T)$

$\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary occupied by a deformable viscoelastic body
$\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ mutually disjoint parts
Γ_C the potential contact surface

We suppose that the body is clamped on Γ_D, so the displacement field vanishes there. Volume forces of density f_1 act in Ω and surface tractions of density f_2 are applied on Γ_N.

Problem formulation

Let $y\colon Q \longrightarrow \mathbb{R}^d$ be the displacement vector, $\sigma\colon Q \longrightarrow \mathbb{S}^d$ the stress tensor and $\varepsilon(y) = (\varepsilon_{ij}(y)) = \frac{1}{2}(y_{i,j} + y_{j,i})$ the linearized strain tensor, where $i, j = 1, \ldots, d$. We employ the viscoelastic constitutive relation

$$\sigma(t) = \mathcal{A}(t, \varepsilon(y'(t))) + \mathcal{B}(\varepsilon(y(t))) + \int_0^t \mathcal{C}(t-s)\varepsilon(y(s))\, ds$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are given constitutive functions. The contact problem can be stated as follows

$$\begin{cases} y''(t) - \operatorname{div}\sigma(y(t), y'(t)) = f_1(t) & \text{in } Q \\ \sigma(t) = \mathcal{A}(t, \varepsilon(y'(t))) + \mathcal{B}(\varepsilon(y(t))) + & \\ \qquad + \int_0^t \mathcal{C}(t-s)\varepsilon(y(s))\, ds & \text{in } Q \\ y = 0 & \text{on } \Gamma_D \times (0,T) \\ \sigma n = f_2 & \text{on } \Gamma_N \times (0,T) \\ -\sigma_\nu(t) \in \partial j_\nu(x, t, y_n u'(t)) & \text{on } \Gamma_C \times (0,T) \\ -\sigma_\tau(t) \in \partial j_\tau(x, t, y'_\tau(t)) & \text{on } \Gamma_C \times (0,T) \\ y(0) = y_0, \;\; y'(0) = y_1 & \text{in } \Omega. \end{cases}$$

Clarke subdifferential

Given a locally Lipschitz function $h \colon X \to \mathbb{R}$, where X is a Banach space, we define (Clarke (1983)):

- **the generalized directional derivative** of h at $x \in X$ in the direction $v \in X$ by

$$h^0(x;v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{h(y+\lambda v) - h(y)}{\lambda}.$$

- **the generalized gradient** of h at x by $\partial h(x)$, is a subset of a dual space X^* given by

$$\partial h(x) = \{\zeta \in X^* : h^0(x;v) \geq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X\}.$$

The locally Lipschitz function h is called **regular** (in the sense of Clarke) at $x \in X$ if for all $v \in X$ the one-sided directional derivative $h'(x;v)$ exists and satisfies $h^0(x;v) = h'(x;v)$ for all $v \in X$.

Example

Given $\beta \in L^\infty_{loc}(\mathbb{R})$ we define the multivalued map $\widehat{\beta}\colon \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ which is obtained from β by filling in the jump procedure as follows

$$\widehat{\beta}(\xi) = [\underline{\beta}(\xi), \overline{\beta}(\xi)] \subset \mathbb{R},$$

where

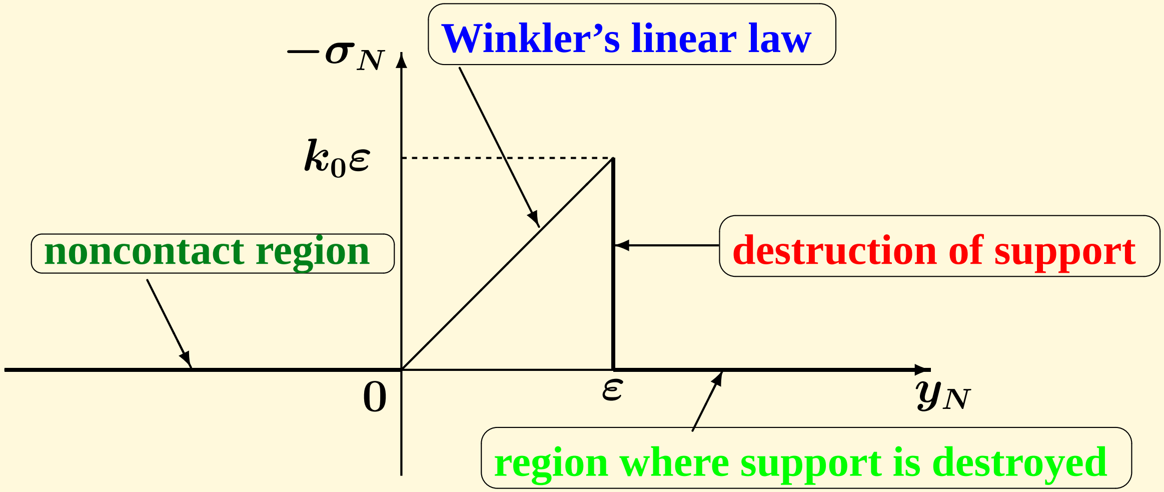
$$\underline{\beta}(\xi) = \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-\xi|\le\delta} \beta(t), \quad \overline{\beta}(\xi) = \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|t-\xi|\le\delta} \beta(t).$$

From Chang (1981), it is known that there exists a locally Lipschitz function $j\colon \mathbb{R} \longrightarrow \mathbb{R}$ determined (up to an additive constant) by the relation $j(t) = \int_0^t \beta(s)\,ds$ and

$$\partial j(t) \subset \widehat{\beta}(t).$$

Additionaly, if $\lim_{t\to\xi^\pm} \beta(t)$ exist for every $\xi \in \mathbb{R}$, then we have

$$\partial j(t) = \widehat{\beta}(t) \text{ for } t \in \mathbb{R}.$$



Nonmonotone diagram for the Winkler-type support

Taking the above into consideration, the boundary condition given by Winkler's law can be written as

$$-\boldsymbol{\sigma}_\nu \in \partial j(y_\nu) = \widehat{\beta}(y_\nu) \text{ on } \Gamma_C,$$

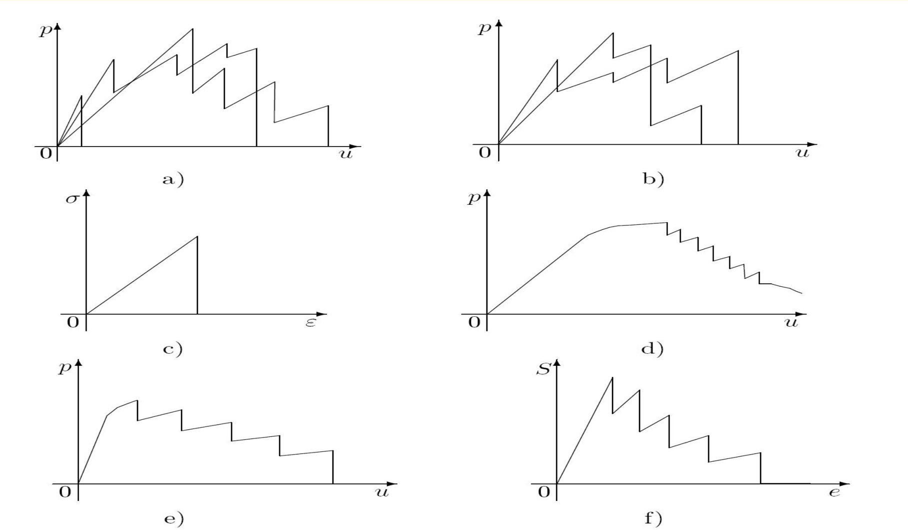
where $j \colon \mathbb{R} \longrightarrow \mathbb{R}$ is of the form

$$j(t) = \int_0^t \beta(s)\, ds = \begin{cases} 0 & \text{if } \; t < 0 \\ \frac{1}{2}\, k_0\, t^2 & \text{if } \; 0 \le t \le \varepsilon \\ \frac{1}{2}\, k_0\, \varepsilon^2 & \text{if } \; t > \varepsilon. \end{cases}$$

with
$\beta \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\beta(t) = \begin{cases} 0 & \text{if } \; t < 0 \\ k_0 t & \text{if } \; t \in [0, \varepsilon) \\ 0 & \text{if } \; t \ge \varepsilon \end{cases}$$

Nonmonotone laws



a) Force-displacement diagrams for laminated products

b) Force-displacement diagrams for glass fiber-reinforced epoxy laminated composites

c) Ply stress-strain curve in a lamina with brittle behavior

d) Force-displacement diagram for a graphite-epoxy composite laminate

e) Force-displacement diagram for an aluminium-beryllium composite beam

f) Scanlon's diagram

Variational formulation

We introduce the spaces $V = \{v \in H^1(\Omega;\mathbb{R}^d) : v = 0 \text{ on } \Gamma_D\}$, $H = L^2(\Omega;\mathbb{R}^d)$, $\mathcal{H} = L^2(\Omega;\mathbb{S}^d)$, $\mathbb{S}^d$ is the space of symmetric matrices of order d.
Let $A\colon (0,T) \times V \to V^*$, $B\colon V \to V^*$ and $C\colon (0,T) \times V \to V^*$ be defined by

$$\langle A(t,u), v\rangle_{V^*\times V} = \langle \mathcal{A}(t,\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}} \text{ for } u, v \in V,\ t \in (0,T),$$

$$\langle Bu, v\rangle_{V^*\times V} = \langle \mathcal{B}(\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}} \text{ for } u, v \in V,$$

$$\langle C(t)u, v\rangle_{V^*\times V} = \langle \mathcal{C}(t)\,\varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}} \text{ for } u, v \in V,\ t \in (0,T).$$

We consider a function $f\colon (0,T) \to V^*$ given by

$$\langle f(t), v\rangle_{V^*\times V} = \langle f_1(t), v\rangle_H + \langle f_2(t), v\rangle_{L^2(\Gamma_N;\mathbb{R}^d)}$$

for all $v \in V$ and a.e. $t \in (0,T)$.

Variational formulation

From the equation of motion and the Green formula, we get

$$\langle \boldsymbol{y}''(t), \boldsymbol{v}\rangle + \langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{H}} = \langle \boldsymbol{f}_1(t), \boldsymbol{v}\rangle_H + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot \boldsymbol{v}\, d\Gamma.$$

Taking into account the boundary conditions, we have

$$\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot \boldsymbol{v}\, d\Gamma = \int_{\Gamma_N} \boldsymbol{f}_2(t)\cdot \boldsymbol{v}\, d\Gamma + \int_{\Gamma_C} \left(\boldsymbol{\sigma}_\nu(t) v_\nu + \boldsymbol{\sigma}_\tau(t)\cdot \boldsymbol{v}_\tau\right)\, d\Gamma$$

From the definition of the Clarke subdifferential, we have

$$-\boldsymbol{\sigma}_\nu(t)\xi \le j_\nu^0(x, t, y_\nu'(x,t); \xi) \quad \text{for all } \xi \in \mathbb{R},$$
$$-\boldsymbol{\sigma}_\tau(t)\cdot \boldsymbol{\eta} \le j_\tau^0(x, t, \boldsymbol{y}_\tau'(x,t); \boldsymbol{\eta}) \quad \text{for all } \boldsymbol{\eta} \in \mathbb{R}^d.$$

We obtain the following hemivariational inequality formulation:

find $y\colon (0,T) \to V$ such that $y \in \mathcal{V}$, $y' \in \mathcal{W}$ and

$$\begin{cases} \langle y''(t) + A(t, y'(t)) + By(t) + \int_0^t C(t-s)y(s)\,ds, v\rangle_{V^*\times V} + \\ + \int_{\Gamma_C} \left(j_\nu^0(x,t,y_\nu'(x,t); v_\nu(x)) + j_\tau^0(x,t,y_\tau'(x,t); v_\tau(x))\right) d\Gamma \geq \\ \qquad \geq \langle f(t), v\rangle_{V^*\times V} \text{ for all } v \in V \text{ and a.e. } t \\ y(0) = y_0, \ \ y'(0) = y_1 \end{cases}$$

or equivalently

$$\begin{cases} \langle y''(t) + A(t, y'(t)) + By(t) + \int_0^t C(t-s)y(s)\,ds, v\rangle_{V^*\times V} + \\ + J^0(t, \gamma y'(t); \gamma v) \geq \langle f(t), v\rangle_{V^*\times V} \text{ for all } v \in V, \text{ a.e. } t \\ y(0) = y_0, \ \ y'(0) = y_1 \end{cases}$$

where $J\colon (0,T)\times X \to \mathbb{R}$, $J(t,v) = \int_{\Gamma_C} j(x,t,v(x))\,d\Gamma$ with $j(x,t,v) = j_\nu(x,t,v_\nu) + j_\tau(x,t,v_\tau)$.

Abstract history-dependent evolution inclusion

PROBLEM $\boldsymbol{P}$: find $\boldsymbol{y} \in \boldsymbol{\mathcal{V}}$ with $\boldsymbol{y}' \in \boldsymbol{\mathcal{W}}$ such that

$$\begin{cases} \boldsymbol{y''(t) + A(t, y'(t)) + By(t) + \mathcal{S}y(t) +} \\ \qquad\qquad \boldsymbol{+ \gamma^* \partial J(t, \gamma y'(t)) \ni f(t) + E(t)u(t)} \text{ a.e. } \boldsymbol{t} \\ \boldsymbol{y(0) = y_0, \;\; y'(0) = y_1.} \end{cases}$$

DEFINITION. A function $\boldsymbol{y} \in \boldsymbol{\mathcal{V}}$ is called a solution of Problem $\boldsymbol{P}$ if and only if $\boldsymbol{y}' \in \boldsymbol{\mathcal{W}}$ and there exists $\boldsymbol{\zeta} \in \boldsymbol{\mathcal{Z}^*}$ such that

$$\begin{cases} \boldsymbol{y''(t) + A(t, y'(t)) + By(t) + \mathcal{S}y(t) + \zeta(t) =} \\ \qquad\qquad\qquad \boldsymbol{= f(t) + E(t)u(t)} \text{ a.e. } \boldsymbol{t} \\ \boldsymbol{\zeta(t) \in \gamma^* \partial J(t, \gamma y'(t))} \text{ a.e. } \boldsymbol{t} \\ \boldsymbol{y(0) = y_0, \;\; y'(0) = y_1.} \end{cases}$$

Hypotheses

$\underline{H(A)}$: $A\colon (0,T)\times V \longrightarrow V^*$ is such that

(i) $A(\cdot, v)$ is measurable on $(0,T)$ for all $v\in V$;

(ii) $A(t,\cdot)$ is strongly monotone, i.e. $\langle A(t,w)-A(t,v), w-v\rangle \geq m_1\|w-v\|_V^2$ for all $w, v\in V$, a.e. $t\in(0,T)$ with $m_1>0$;

(iii) $\|A(t,v)\|_{V^*}\leq a(t)+b\|v\|_V$ for all $v\in V$, a.e. $t\in(0,T)$ with $a\in L^2(0,T)$, $a\geq 0$, $b>0$;

(iv) $\langle A(t,v),v\rangle \geq \alpha\|v\|_V^2$ for all $v\in V$, a.e. $t\in(0,T)$ with $\alpha>0$.

$\underline{H(B)}$: $B\colon V\longrightarrow V^*$ is bounded, linear, monotone and symmetric.

$\underline{H(S)}$: $\mathcal{S}\colon\mathcal{V}\longrightarrow\mathcal{V}^*$ is such that $\exists L_S>0$:

$$\|\mathcal{S}y_1(t)-\mathcal{S}y_2(t)\|_{V^*}\leq L_S\int_0^t\|y_1(s)-y_2(s)\|_V\,ds$$

$$\forall\, y_1,\ y_2\in\mathcal{V},\ \text{a.e.}\ t\in(0,T).$$

$\underline{H(E)}$: $E\in L^\infty(0,T;\mathcal{L}(Y,V^*))$ and Y is a separable reflexive Banach space (the space of controls).

Hypotheses

$\underline{H(J)}$: $J\colon (0,T) \times L^2(\Gamma_C;\mathbb{R}^d) \to \mathbb{R}$ is a functional such that

(i) $J(\cdot, v)$ is measurable for all $v \in L^2(\Gamma_C;\mathbb{R}^d)$;

(ii) $J(t,\cdot)$ is locally Lipschitz for a.e. $t \in (0,T)$;

(iii) $\|\partial J(t,v)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leq c_0 \left(1 + \|v\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right)$ for all $v \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$ with $c_0 > 0$;

(iv) $J^0(t,v;-v) \leq d_0 \left(1 + \|v\|_{L^2(\Gamma_C;\mathbb{R}^d)}\right)$ for all $v \in L^2(\Gamma_C;\mathbb{R}^d)$, a.e. $t \in (0,T)$ with $d_0 \geq 0$;

(v) $(z_1 - z_2, w_1 - w_2)_{L^2(\Gamma_C;\mathbb{R}^d)} \geq -m_2\|w_1 - w_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}$ for all $z_i \in \partial J(t,w_i)$, $w_i \in L^2(\Gamma_C;\mathbb{R}^d)$, $i = 1,2$, a.e. $t \in (0,T)$ with $m_2 \geq 0$.

$\underline{(H_0)}$: $f \in \mathcal{V}^*$, $y_0 \in V$, $y_1 \in H$.

$\underline{(H_1)}$: $m_1 > m_2 \|\gamma\|^2 c_e^2$, where $\|\gamma\| = \|\gamma\|_{\mathcal{L}(Z,L^2(\Gamma;\mathbb{R}^d))}$ and $c_e > 0$ is an embedding constant of V into Z.

Existence and uniqueness result

THEOREM 1. Assume that $H(A)$, $H(B)$, $H(S)$, $H(E)$, $H(J)$, (H_0), (H_1) hold and $u \in \mathcal{U} = L^2(0,T;Y)$. Then Problem P admits a unique solution.

Idea of the proof:

- existence of unique solution for evolutionary inclusion without memory term
  - regular initial data (i.e. $y_1 \in V$)
  - a standard reduction technique – an evolution inclusion of the first order
  - a surjectivity result for the sum of two operators: one operator is closed, densely defined and maximal monotone, and the second one is bounded, coercive and pseudomonotone w.r.t. the graph norm topology of the domain of the first operator
  - remove the restriction on the initial data a density argument
- a fixed point argument

(Theorem 2.1 in [Migorski, O., Sofonea, M^3AS 2008]).

□

Continuous dependence on a control variable

PROPOSITION Under hypotheses of Theorem 1 and the condition

$$\underline{(H_2)}: \quad \alpha > 2T\sqrt{T}\|C\|_{L^2(0,T;\mathcal{L}(V,V^*))},$$

if $\{u_n\} \subset \mathcal{U}$, $u_n \longrightarrow u$ weakly in $\mathcal{U}$, then $y_n \longrightarrow y$ weakly in $\mathcal{V}$ and $y_n' \longrightarrow y'$ weakly in $\mathcal{W}$, where y_n and y are unique solutions of Problem P corresponding to u_n and u, respectively.

Idea of the proof:
a priori estimate

$$\|y_n\|_{C(0,T;V)} + \|y_n'\|_{\mathcal{W}} \leq \overline{c}\,(1 + \|y_0\|_V + \|y_1\|_H + \|f\|_{\mathcal{V}^*} + \|u_n\|_{\mathcal{U}})$$

with $\overline{c} > 0$ independent on n. □

A solution map

$$S\colon \mathcal{U} \ni u \mapsto S(u) \subset \mathcal{X},$$

where $\mathcal{X} = \{y \in \mathcal{V} : y' \in \mathcal{W}\}$, has a closed graph in (w-$\mathcal{U}$) $\times$ (w-$\mathcal{X}$) - topology.

Bolza optimal control problem for hvi

(CP)

$$\begin{cases} \Phi(y,u) = l(y(T), y'(T)) + \int_0^T F\left(t, y(t), y'(t), u(t)\right)\, dt \\ \Phi(y,u) \longrightarrow \inf =: m \\ \\ \text{where } u(t) \in U(t) \text{ a.e. } t \in (0,T), u(\cdot) \text{ is measurable, } y \in S(u) \end{cases}$$

$S(u)$ is the set of solutions of Problem P corresponding to a control u.

H(Φ): $l \colon H \times H \longrightarrow \mathbb{R}$ is weakly lsc;

$F \colon [0,T] \times H \times H \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is measurable and:

(i) $F\left(t,\cdot,\cdot,\cdot\right)$ is seq. lsc on $H \times H \times Y$ for a.e. $t \in (0,T)$,

(ii) $F\left(t,y,z,\cdot\right)$ is convex on Y, for all $y,z \in H$ and a.e. t,

(iii) there exist $M > 0$, $\psi \in L^1(0,T)$ s.t. for all $y,z \in H$, $u \in Y$ and a.e. $t \in (0,T)$, we have

$$F(t,y,z,u) \geq \psi(t) - M\left(\|y\|_H + \|z\|_H + \|u\|_Y\right).$$

H(U): $U\colon [0,T] \longrightarrow 2^Y \setminus \{\emptyset\}$ is a multifunction s.t. for all $t \in [0,T]$, $U(t)$ is a closed convex subset of Y and $t \mapsto \|U(t)\|_Y \in L^\infty_+$.

$S^q_U = \{w \in \mathcal{U} = L^q(0,T;Y) : w(t) \in U(t) \text{ a.e. } t\}$ is nonempty [Hu, Papageorgiou (1997)].

By an admissible state–control pair (y,u) for (CP) we understood a pair of a state function y (which solves Problem P) and a control function $u \in S^2_U$. An admissible pair (y,u) is called an optimal solution to (CP) if and only if $\Phi(y,u) = m$.

Existence of optimal solution

THEOREM 2. If the hypotheses $H(A)$, $H(B)$, $H(J)$, (H_0), (H_1), (H_2), $H(C)$, $H(\Phi)$ and $H(U)$ hold, then the problem (CP) admits an optimal solution.

Proof. Applying the direct method of the calculus of variations.
By Theorem 1, $S(u) \neq \emptyset$ for all fixed $u \in \mathcal{U}$.
Let $\{(y_n, u_n)\} \subseteq \mathcal{X} \times \mathcal{U}$ be a minimizing sequence of admissible state–control pairs for the problem (CP), i.e. $y_n \in S(u_n)$, $u_n(t) \in U(t)$ for a.e. $t \in (0, T)$ and $\lim_{n \to +\infty} \Phi(y_n, u_n) = m$.
From the hypothesis $H(U)$, $u_n \longrightarrow u$ weakly in $\mathcal{U}$ and $u(t) \in U(t)$ for a.e. t.
From the a priori estimate, we get $\|y_n\|_{\mathcal{V}} \leq c_1$, $\|y_n'\|_{\mathcal{W}} \leq c_2$ with $c_1, c_2 > 0$ independent of n. So for a subsequence we have

$$y_n \longrightarrow y \quad \text{weakly in } \mathcal{V} \quad \text{and} \quad y_n' \longrightarrow y' \quad \text{weakly in } \mathcal{W}.$$

From Proposition 2 $y \in S(u)$. Hence (y, u) is an admissible state–control pair.

It is also an optimal solution.
In fact, using the compactness of the embedding $\mathcal{W} \subset \mathcal{H}$, for a next subsequence, we have

$$y_n \longrightarrow y \quad \text{and} \quad y_n' \longrightarrow y' \quad \text{both in } \mathcal{H}.$$

Invoking now Theorem 2.1 of Balder (1987) we obtain that the cost functional Φ is sequentially lower semicontinuous on $L^2(0,T;H \times H) \times (w - L^q(0,T;Y))$. So,

$$m \leq \Phi(y,u) \leq \liminf_{n \to +\infty} \Phi(y_n, u_n) = m,$$

which proves the theorem. □

Remark

Analogously to Theorem 2, under the hypotheses $\boldsymbol{H(A)}$, $\boldsymbol{H(B)}$, $\boldsymbol{H(J)}$, $(\boldsymbol{H_0})$, $(\boldsymbol{H_1})$, $\boldsymbol{H(C)}$ and $\boldsymbol{H(\Phi)}$, we can establish the existence result for the optimal control problem of the form

$$\inf\{\Phi(y,u) : y \in S(u), u \in U_{ad}\},$$

where U_{ad} is a nonempty, weakly compact subset of $\mathcal{U}$ (a set of admissible controls). In this case having a minimizing sequence $\{(y_n, u_n)\}$, $y_n \in S(u_n)$, $u_n \in U_{ad}$, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } \mathcal{U} \quad \text{and} \quad u \in U_{ad}.$$

Then we proceed as in the proof of Theorem 2.

Example of a cost functional

The hypothesis $H(\Phi)$ incorporates the quadratic cost functionals studied by Lions (1971) and by Ha and Nakagiri (1997). In particular, we may take a combination of these functionals, namely

$$\begin{aligned}\Phi(y,u) &= \varrho_1 \, \|y(T) - y_d\|_H^2 + \varrho_2 \, \|y'(T) - \overline{y_d}\|_H^2 + \\ &+ \varrho_3 \int_0^T \|\mathcal{O}_1 y(t) - z_d(t)\|_{\mathcal{H}}^2 \, dt + \\ &+ \varrho_4 \int_0^T \|\mathcal{O}_2 y'(t) - \overline{z_d}(t)\|_{\mathcal{H}}^2 \, dt + \\ &+ \varrho_5 \int_0^T \langle Ru(t), u(t)\rangle_{Y^* \times Y} \, dt,\end{aligned}$$

where $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{L}(H)$ are observation operators, $R \in \mathcal{L}(Y, Y^*)$ is a positive defined and symmetric operator on Y, $y_d, \overline{y_d} \in H$, $z_d, \overline{z_d} \in \mathcal{H}$ are given elements (desired outputs) and $\varrho_i \geq 0$ $(i = 1, \ldots, 5)$ are some constants (weights).

A convergence result for inclusions

$\underline{H(C)_\varepsilon}$: C, $C_\varepsilon \in L^2(0,T;\mathcal{L}(V,V^*))$ and $C_\varepsilon \to C$ in $L^2(0,T;\mathcal{L}(V,V^*))$.

PROBLEMs P_ε: find $y_\varepsilon \in \mathcal{V}$ such that $y'_\varepsilon \in \mathcal{W}$ and

$$\begin{cases} y''_\varepsilon(t) + A(t, y'_\varepsilon(t)) + By_\varepsilon(t) + \int_0^t C_\varepsilon(t-s)y_\varepsilon(s)\,ds + \\ \qquad + \gamma^*\partial J(t, \gamma y'(t)) \ni f(t) + E(t)u(t) \text{ a.e. } t \\ y_\varepsilon(0) = y_0, \ \ y'_\varepsilon(0) = y_1. \end{cases}$$

THEOREM 3. Assume that $H(A)$, $H(B)$, $H(C)_\varepsilon$, $H(E)$, $H(J)$, (H_0), (H_1), (H_2) hold and $u \in \mathcal{U}$. Then, the unique solution y_ε of Problems P_ε converges to the solution y of Problem P, i.e.

$$\lim_{\varepsilon\to 0}\left(\|y_\varepsilon - y\|_{C(0,T;V)} + \|y'_\varepsilon - y'\|_{C(0,T;H)} + \|y'_\varepsilon - y'\|_{\mathcal{V}}\right) = 0.$$

Vanishing relaxation operator

THEOREM 4. Assume the hypotheses of Theorem 3 and let $y_\varepsilon \in \mathcal{V}$ with $y'_\varepsilon \in \mathcal{W}$ be a unique solution of the problem

$$\begin{cases} y''_\varepsilon(t) + A(t, y'_\varepsilon(t)) + B(t, y_\varepsilon(t)) + \varepsilon \int_0^t C(t-s) y_\varepsilon(s)\, ds + \\ + \gamma^* \partial J(t, \gamma y'_\varepsilon(t)) \ni f(t) + E(t)u(t) \text{ a.e. } t \\ y_\varepsilon(0) = y_0, \;\; y'_\varepsilon(0) = y_1. \end{cases}$$

for $\varepsilon > 0$. Then, y_ε converges to y in the following sense

$$\lim_{\varepsilon \to 0} \left(\|y_\varepsilon - y\|_{C(0,T;V)} + \|y'_\varepsilon - y'\|_{C(0,T;H)} + \|y'_\varepsilon - y'\|_{\mathcal{V}} \right) = 0,$$

where $y \in \mathcal{V}$ with $y' \in \mathcal{W}$ is the unique solution of the problem

$$\begin{cases} y''(t) + A(t, y'(t)) + By(t) + \gamma^* \partial J(t, \gamma y'(t)) \ni f(t) + E(t)u(t) \\ y(0) = y_0, \;\; y'(0) = y_1. \end{cases}$$

Proof. $C_\varepsilon = \varepsilon C$.

□

Asymptotic behavior of optimal solutions

$(\mathrm{CP})_\varepsilon$

$$\begin{cases} \Phi(y_\varepsilon, u) = l(y_\varepsilon(T), y'_\varepsilon(T)) + \int_0^T F\left(t, y_\varepsilon(t), y'_\varepsilon(t), u(t)\right)\, dt \\ \Phi(y_\varepsilon, u) \longrightarrow \inf =: m_\varepsilon \\ \\ \text{where } u(t) \in U(t) \text{ a.e. } t, u(\cdot) \text{ is measurable, } y_\varepsilon \in S_\varepsilon(u). \end{cases}$$

$S_\varepsilon(u)$ is the set of solutions of Problem P_ε corresp. to a control u.

THEOREM 5. Assume hypotheses $H(A)$, $H(B)$, $H(C)$, $H(E)$, $H(J)$, (H_0), (H_1), (H_2), $H(\Phi)$ and $H(U)$. Then

(1) for every $\varepsilon > 0$, the control problem $(\mathrm{CP})_\varepsilon$ has at least one optimal solution $(u^*_\varepsilon, y^*_\varepsilon)$ with minimal value $m_\varepsilon = \Phi(u^*_\varepsilon, y^*_\varepsilon)$;

(2) there exists a subsequence of $\{(u^*_\varepsilon, y^*_\varepsilon)\}$ which converges weakly to (u^*, y^*) is an optimal solution to (CP).

(3) We have the convergence of minimal values $m_\varepsilon \longrightarrow m$, as $\varepsilon \longrightarrow 0$.

Summary

- the Bolza distributed parameter control problem
- necessary condition $(H(\Phi))$ for existence of optimal solution to (CP)
- other possibility: the time optimal control, the maximum stay control problems ect.
- our efforts are of importance in the development of control theory for a large class of mechanical and engineering problems involving nonmonotone and multivalued relations

References

[1] S. Migorski, A. Ochal, M. Sofonea, *Integrodifferential hemivariational inequalities with applications to viscoelastic frictional contact*, *Mathematical Models and Methods in Applied Sciences* **18** (2008), 271–290.

Problem: *Find* $y \in \mathcal{V}$ *with* $y' \in \mathcal{W}$ *such that*

$$y''(t) + A(t, y'(t)) + By(t) + \int_0^t C(t-s)y(s)\,ds + \\ + \gamma^* \partial J(t, \gamma y(t)) \ni f(t) \text{ a.e. } t \in (0, T),$$

$$y(0) = y_0,\ y'(0) = y_1.$$

[2] S. Migorski, A. Ochal, M. Sofonea, *History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics*, *Nonlineal Analysis RWA* **12** (2011), 3384–3396.

Problem: *Find* $y \in \mathcal{V}$ *such that*

$$A(t, y(t)) + \mathcal{S}y(t) + \gamma^* \partial J(t, \gamma y(t)) \ni f(t) \text{ a.e. } t \in (0, T).$$