ON OPERATORS WITH $C_{.0}$ NONUNITARY PART

PATRYK PAGACZ

By $C_{.0}$ contraction we mean a contraction $T \in \mathcal{B}(\mathcal{H})$ such that its adjoint is strongly stable ($\|T^{*n}x\| \to 0$ for all $x \in \mathcal{H}$). A bounded operator T is called hyponormal if $T^*T - TT^*$ is a positive operator (equivalent: $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$).

Since pioneering papers by Putnam where it was proved that nonunitary part of hyponormal contraction is $C_{.0}$, Duggal and Kubrulsy extended this result. We show this property for further generalization of hyponormal contractions.

In our presentation we consider the following classes of operators (which extend class of hyponormal operators):

Definition 1. *$T \in \mathcal{B}(\mathcal{H})$ is said to be* (p,k)-quasihyponormal *if*

$$T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$$

for a positive number $0 < p \leq 1$ and a positive integer k.

Definition 2. *$T \in \mathcal{B}(\mathcal{H})$ is said to be* $k*$-paranormal *if*

$$\|T^*x\|^k \leq \|T^k x\| \|x\|^{k-1}$$

for each $x \in \mathcal{H}$.

Firstly, we present a new characterization of contraction with nonunitary part. Let the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $Tx_{n+1} = x_n$ be called a backward sequence of T. Then we prove the following.

Theorem 1. *Let T be a contraction. The following conditions are equivalent:*

(1) *for any bounded backward sequence $\{x_n\}_{n\in\mathbb{N}}$ of T, the sequence of norms $\{\|x_n\|\}_{n\in\mathbb{N}}$ is constant,*
(2) *nonunitary part of T is of class $C_{.0}$.*

Secondary, from above criterium we show that $k*$-paranormal and (p,k)-quasihyponormal operators have $C_{.0}$ nonunitary part.

Finally, we characterize the set
$\{x \in \mathcal{H} | \exists \{x_n\}_{n\in\mathbb{N}}$ bounded backward sequence, $x = x_0\}$ as the image of operator which the is root of strong limit of the sequence $\{T^nT^{*n}\}_n$.

References

[1] B.P. Duggal, *On unitary parts of contractions*, Indian J. Pure Appl. Math. 25(1994), 1243-1227.
[2] B.P. Duggal, *On characterising contractions with C_{10} pure part*, Integral Equations Operator Theory 27(1997), 314-323.
[3] B.P. Duggal and C.S. Kubrusly, *Paranormal contractions have property PF*, Far East J. Math. 14(2004), 237-249.
[4] B.P. Duggal, I.H. Jeon and I.H. Kim, *On $*$-paranormal contractions and properties for $*$-class $\mathcal{A}$ operators*, 2011.
[5] E. Durszt, *Contractions as restricted shifts*, Acta Sci. Math. (Szeged) 48(1985), 129-134.
[6] C.S. Kubrusly, P.C.M. Vieira and D.O. Pinto, *A Decomposition for a class of contractions*, Advan. Math. Sci. Appl. 2(1993), 335-343.
[7] C.S. Kubrusly and P.C.M. Vieira, *Strong stability for cohyponormal operators*, J. Operator Theory 31(1994), 123-127.
[8] P. Pagacz, *On Wold-type decomposition*, http://ssdnm.mimuw.edu.pl/student/35
[9] C.R. Putnam, *Hyponormal contractions and strong power convergence*, Pacific J. Math. 57(1975), 531-538.
[10] Z. Sebestyen, *On range of adjoint operators in Hilbert space*, Acta Sci. Math. (Szeged) 46(1983), 295-298.
[11] K. Tanahashi, A. Uchiyama and M. Cho, *Isolated points of spectrum of (p,k)-quasihyponormal operators*, Linear Algebra Appl. 382(2004), 221-229.