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An optimal control problem for an elastic beam with in a dynamic contact with a rigid obstacle

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Abstract: We deal with the optimal control problem governed by a hyperbolic variational inequality describing the perpendicular vibrations of a beam clamped on the left end with a rigid obstacle at the right end. A variable thickness of a beam plays the role of a control parameter.

1 Introduction

The dynamic contact problems are not frequently solved in the framework of variational inequalities. For the elastic problems there is only a very limited amount of results available (cf. [4] and there cited literature). The inner dynamic obstacle problem for the plate with moderately large deflections has been solved in [1]. We deal here with an optimal design problem for an elastic beam in dynamic contact with a rigid obstacle. A variable thickness of a beam plays the role of a control variable. We have solved such problem in [2]with the fixed convex set of admissible deflections expressing the boundary contact. The admissible sets of deflections depends here on the control variable due to the character of the unilateral condition. A similar problem for the stationary elliptic state variational inequality has been considered in [3]. In contrast to it there is no uniqueness result in the dynamic case and hence the minimum will depend both on the thickness as the control and the deflection as the state variable. Solving the state hyperbolic variational inequality we apply the method of penalization in the same way as in [1].

2 Solving of the state problem

2.1 The state problem formulation

We consider an elastic beam of the length $L > 0$. Its variable thickness is expressed by the positive function $x \mapsto 2e(x),\ x \in [0, L]$, the constant $d > 0$ involves the material and geometrical characteristics. We assume for simplicity $\rho \equiv 1$ the density of the material and the beam free on the both ends. The rigid inner obstacle is characterized by the function $\Phi : [0, L] \mapsto \mathbb{R}$. If f is a perpendicular load acting on the beam, $u_0,\ v_0$ the initial displacement and velocity respectively, then the vertical displacement u solves the following hyperbolic initial-boundary value problem with an unknown contact force g and the complementary conditions.

$$e(x)\ddot{u} + d(e^3(x)u_{xx})_{xx} = f(t, x) + g(t, x) \text{ in } (0, T) \times (0, L), \tag{1}$$

$$u_{xx}(t, 0) = [e^3(x)u_{xx}]_x(t, 0) = u_{xx}(t, L) = [e^3(x)u_{xx}]_x(t, L) = 0,\ t \in (0, T], \tag{2}$$

$$u \geq \Phi + \frac{1}{2}e,\ g \geq 0,\ (u - \Phi - \frac{1}{2}e)g = 0,\ \text{in } (0, T) \times (0, L], \tag{3}$$

$$u(0, x) = u_0(x),\ \dot{u}(0, x) = v_0(x),\ x \in (0, L). \tag{4}$$

We set $I = (0, T),\ Q = I \times (0, L)$. and introduce the Hilbert spaces

$$H \equiv L_2(0, L),\ V \equiv H^2(0, L) = \{y \in L_2(0, L) :\ y'' \in L_2(0, L)\}$$

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with the inner products and the norms

$$(y,z)=\int_0^L y(x)z(x)\,dx,\ \ |y|_0=(y,y)^{1/2},\ \ y,\,z\in H;$$

$$((y,z))=\int_0^L [y(x)z(x)+y''(x)z''(x)]\,dx,\ \ \|y\|=((y,y))^{1/2},\ \ y,\,z\in V.$$

Further we set $\mathcal{V}=L_\infty(I,V)$,

$$\mathcal{K}(e)=\{y\in\mathcal{V}:\ y(t,x)\geq \Phi(x)+\frac{1}{2}e(x)\ \text{for a.e.}\ \ (t,x)\in(0,T]\times[0,L]\},$$
$$K(e)=\{v\in V:\ v(x)\geq\Phi(x)+\frac{1}{2}e(x)\ \text{for all}\ x\in[0,L]\}$$

and assume

$$f\in L_2(Q),\ \Phi\in V,\ u_0\in K(e),\ v_0\in L_2(\Omega);\ e\in H^2[0,L],\ 0<e_1\leq e\leq e_2.$$

Definition 2.1 *Function $u\in\mathcal{K}(e)$ is a weak solution of the problem* (1)-(4) *if $\ddot{u}\in\mathcal{V}^*$, the initial conditions* (4) *are fulfilled in a certain generalized sense and the inequality*

$$\langle\langle\ddot{u},e(y-u)\rangle\rangle+d\int_Q e^3(x)u_{xx}(y-u)_{xx}\,dt\,dx\geq\int_Q f(x)(y(t,x)-u(t,x))\,dt\,dx \tag{5}$$

holds for any $y\in\mathcal{K}(e)$.

The expression $\langle\langle\cdot,\cdot\rangle\rangle$ means the duality between the spaces $\mathcal{V}^*$ and $\mathcal{V}$ as the extension of the inner product in the space $L_2(Q)$.

2.2 Penalization

We define for $\varepsilon>0$ the *penalized problem* in the variational form:

To find $u_\varepsilon\in\mathcal{V}$ such that $\ddot{u}_\varepsilon\in L_2(I;V^*)$ and

$$\langle\langle\ddot{u}_\varepsilon,ey\rangle\rangle+\int_Q\left[de^3(x)u_{\varepsilon xx}y_{xx}]-\varepsilon^{-1}(u_\varepsilon-\Phi(x)-\frac{1}{2}e(x))^-y\right]dt\,dx=\int_Q fy\,dt\,dx \tag{6}$$
$\forall y\in L_2(I,V)$,

$$u_\varepsilon(0,x)=u_0(x),\ \dot{u}_\varepsilon(0,x)=v_0(x),\ x\in(0,L). \tag{7}$$

We verify the existence of a solution to the penalized problem and useful *a priori* estimates by the Galerkin method.

Theorem 2.2 *There exists a solution $u\equiv u_\varepsilon$ of the problem* (6), (7) *fulfilling the estimate*

$$\|\dot{u}_\varepsilon\|^2_{L_\infty(I,L_2(0,L))}+\|u_\varepsilon\|^2_{L_\infty(I,V)}\leq C(d,e_1,e_2,u_0,v_0,f),$$
$$C(d,e_1,e_2,u_0,v_0,f)=\left(\frac{2}{e_1}+\frac{1}{de_1^3}\right)\left(e_2|v_0|_0^2+de_2^3\|u_0\|^2+\frac{2}{e_1}\|f\|^2_{L_1(I,L_2(0,L))}\right). \tag{8}$$

Proof. Let us denote by $\{w_i\in V;\ i\in\mathbb{N}\}$ a basis of V. We construct the Galerkin approximation u_m of a solution in a form

$$u_m(t)=\sum_{i=1}^m\alpha_i(t)w_i,\ \alpha_i(t)\in\mathbb{R},\ i=1,...,m,\ m\in\mathbb{N},$$

$$\int_0^L\left(e(x)\ddot{u}_m w_i+de^3(x)u_{m_{xx}}w_{i_{xx}}-\varepsilon^{-1}(u_m^- -\Phi(x)-\frac{1}{2}e(x))w_i\right)dx=$$
$$\int_0^L f(t)w_i\,dx,\ i=1,...,m, \tag{9}$$

$$u_m(0)=u_{0m},\ \dot{u}_m(0)=v_{0m},\ u_{0m}\to u_0\ \text{in}\ V\ \text{and}\ v_{0m}\to v_0\ \text{in}\ L_2(0,L). \tag{10}$$

The solution $u_m : [0,T] \mapsto \mathbb{R}$ exists due the existence theory for the 2nd-order system of ordinary differential equations. After multiplying the equation (9) by $\dot{\alpha}_i(t)$, summing up with respect to i and integrating we obtain the estimate

$$\|\dot{u}_m\|^2_{C(\bar{I};L_2(0,L))} + \|u_m\|^2_{C(\bar{I};V)} + \varepsilon^{-1}\|u_m^- - \Phi - \frac{1}{2}e\|^2_{C(\bar{I};L_2(0,L))} \leq c_1. \tag{11}$$

From (9) we obtain directly the estimate

$$\|\ddot{u}_m\|^2_{L_2(I;W_m^*)} \leq c_\varepsilon,\ m \in \mathbb{N}, \tag{12}$$

where W_m is the linear hull of $\{w_i\}_{i=1}^m$.

We proceed with the convergence of the Galerkin approximation. Applying the estimates (11), (12), the density of $\bigcup_{m=1}^\infty W_m$ and the compact imbedding theorem we obtain for a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$) a function $u \in \mathcal{V} \cap H^1(Q)$ with $\ddot{u} \in L_2(I;W^*)$ and the convergences

$$\begin{aligned} &\ddot{u}_m \rightharpoonup \ddot{u} && \text{in } L_2(I;W^*),\\ &\dot{u}_m \rightharpoonup^* \dot{u} && \text{in } L_\infty(I;L_2(0,L)),\\ &u_m \rightharpoonup^* u && \text{in } \mathcal{V},\\ &u_m \to u && \text{in } C(\bar{I};H^{2-\delta}(0,L))\ \forall\delta > 0. \end{aligned} \tag{13}$$

Let $\mu \in \mathbb{N},\ z_\mu = \sum_{i=1}^\mu \phi_i(t)w_i,\ \phi_i \in \mathcal{D}(0,T),\ i = 1,...,\mu$. The convergence process (13) implies

$$\langle\langle \ddot{u}_\varepsilon, e z_\mu\rangle\rangle + \int_Q \left[d e u_{xx} z_{\mu_{xx}} - \varepsilon^{-1}(u_\varepsilon - \Phi - \frac{1}{2}e)^- z_\mu \right] dt\,dx = \int_Q f z_\mu\,dt\,dx$$

Functions $\{z_\mu\}$ form a dense subset of the set $L_2(I;V)$, hence a function $u \equiv u_\varepsilon$ fulfils the identity (6) and $\ddot{u} \in L_2(I;V^*)$. The initial conditions (7) follow due to (10) and the proof of the existence of a solution is complete.

In order to achieve the *a priori* estimate (8) we put

$$y = \begin{cases} \dot{u_\varepsilon} \text{ for } t \leq s \\ 0 \text{ for } t > s \end{cases}$$

in (6) with an arbitrary $s \in I$. After performing the integration we obtain the inequalities

$$\begin{aligned} e_1|\dot{u}_\varepsilon|_0^2(s) + d e_1^3\|u_\varepsilon\|^2(s) &\leq e_2|v_0|_0^2 + d e_2^3\|u_0\|^2 + 2\int_0^s (f,\dot{u}_\varepsilon)(t)\,dt \\ &\leq e_2|v_0|_0^2 + d e_2^3\|u_0\|^2 + \frac{2}{e_1}\|f\|^2_{L_1(I;L_2(0,L))} + \frac{1}{2}e_1\|\dot{u}_\varepsilon\|^2_{L_\infty(I,L_2(0,L))}\ \forall s \in I \end{aligned}$$

and the estimate (8) follows.

2.3 The limit process to the original state problem

The *a priori* estimates and the convergence process derived in the previous section imply the estimate

$$\|\dot{u}_\varepsilon\|^2_{L_\infty(I,L_2(0,L))} + \|u_\varepsilon\|^2_{L_\infty(I,V)} + \varepsilon^{-1}\|u_\varepsilon^- - \Phi - \frac{1}{2}e\|^2_{C(\bar{I},L_2(0,L))} \leq c_2. \tag{14}$$

Let us set $y(t,x) \equiv 1$ in (6). The estimate (14) implies the estimates

$$0 \leq \varepsilon^{-1}\int_Q (u_\varepsilon - \Phi - \frac{1}{2}e)^-\,dt\,dx \leq c_3,\ \|\ddot{u}_\varepsilon\|_{L_1(I;V^*)} \leq c_4. \tag{15}$$

Then there exists a sequence $\varepsilon_n \searrow 0$, a function $u \in \mathcal{V} \cap H^1(Q)$ and a functional $g \in (L_\infty(Q))^*$ such that $\ddot{u} \in (L_\infty(I,V))^*$, $\dot{u} \in L_\infty(I, L_2(0,L)) \cap C_w(\bar{I}, L_2(0,L))$ and for $u_n \equiv u_{\varepsilon_n}$ the following convergences hold

$$\begin{aligned}
\ddot{u}_n &\rightharpoonup^* \ddot{u} && \text{in } (L_\infty(I;V))^* \\
\dot{u}_n &\rightharpoonup \dot{u} && \text{in } L_2(I;V), \\
u_n &\rightharpoonup^* u && \text{in } \mathcal{V}, \\
u_n &\to u && \text{in } C(\bar{I}, H^{2-\delta})\ \forall \delta > 0, \\
u_n^- - \Phi - \frac{1}{2}e &\to 0 && \text{in } C(\bar{I}, L_2(\Omega)), \\
\varepsilon_n^{-1}(u_n^- - \Phi - \frac{1}{2}e) &\rightharpoonup^* g && \text{in } (L_\infty(Q))^*.
\end{aligned} \tag{16}$$

Let us define the operator $A(e) : V \mapsto V^*$ by

$$\langle A(e)u, y\rangle_* = d \int_0^L e^3(x) u_{xx} y_{xx}\, dx,\ \ u,\ y \in V. \tag{17}$$

The performed convergences implies that the limit function u fulfils the equation in V^*

$$e\ddot{u} + A(e)u = f + g, \tag{18}$$

where $e\ddot{u} \in \mathcal{V}^*$ is defined by $\langle\langle e\ddot{u}, y\rangle\rangle = \langle\langle \ddot{u}, ey\rangle\rangle\ \forall y \in \mathcal{V}$.

The limit functional g represents a contact force acting between the beam and the obstacle. It fulfils $\langle\langle g, u - \Phi - \frac{1}{2}e\rangle\rangle = 0$ and $\langle\langle g, z\rangle\rangle \geq 0\ \forall z \in V\ z \geq 0$ due to the last convergence in (16) and hence the inequality (5) is fulfilled. The initial condition for a deflection u is fulfilled in the space $H^{2-\delta}(\Omega)$ and the initial velocity id fulfilled in a generalized sense. Hence we have proved

Theorem 2.3 *Let* $u_0 \in K,\ v_0 \in L_2(0,L),\ f \in L_2(Q),\ \Phi \in V,\ e \in V,\ 0 < e_1 \leq e \leq e_2$.

Then there exists a weak solution of the State problem (1)-(4) *fulfilling the estimate*

$$\|\dot{u}\|^2_{L_\infty(I,L_2(0,L))} + \|u\|^2_{L_\infty(I,V)} \leq C(d, e_1, e_2, u_0, v_0, f) \tag{19}$$

with the constant $C(d, e_1, e_2, u_0, v_0, f)$ *defined in* (8).

3 Optimal control problem

We consider a cost functional

$$J : \mathcal{V} \times C^2([0,L]) \mapsto \mathbb{R}^+$$

fulfilling

$$u_n \rightharpoonup u \text{ in } \mathcal{V},\ e_n \to e \text{ in } C^2([0,L]) \Rightarrow J(u,e) \leq \liminf_{n\to\infty} J(u_n, e_n).$$

Let

$$E_{ad} = \{e \in H^3(0,L) :\ 0 < e_1 \leq e(x) \leq e_2\ \forall x \in [0,L],\ \|e\|_{H^3(0,L)} \leq e_3\}$$

be the set of admissible thicknesses. We remark that E_{ad} is compact in $H^2(0,L)$.

Before formulating the Optimal control problem we introduce the space of functions

$$\begin{aligned}\mathcal{W} = \{v \in L_\infty(I; L_2(0,L)) :\ \exists \dot{v} \in L_\infty(I;V)^* \text{ and } \{v_n\} \subset H^1(I; L_2(0,L)) \\ \text{such that } v_n \rightharpoonup^* v \text{ in } L_\infty(I; L_2(0,L)),\ \dot{v}_n \rightharpoonup^* \dot{v} \text{ in } L_\infty(I;V)^*\}.\end{aligned}$$

Optimal control problem $\mathcal{P}$: To find a couple $(u_*, e_*) \in U_{ad}(e_*) \times E_{ad}$ such that

$$J(u_*, e_*) \leq J(u,e)\ \forall (u,e) \in U_{ad}(e) \times E_{ad}, \tag{20}$$

$$\begin{aligned}U_{ad}(e) = \{u \in \mathcal{K}(e) :\ \dot{u} \in \mathcal{W},\ u \text{ is a weak solution of } (1) - (4), \\ \|\dot{u}\|^2_{L_\infty(I;L_2(0,L))} + \|u\|^2_{L_\infty(I;V)} \leq C_1\}\end{aligned} \tag{21}$$

with $C_1 \geq C(d, e_1, e_2, u_0, v_0, f)$ - a positive constant defined in (8).

The construction of a solution $u \in \mathcal{K}(e)$ using the penalization method in Theorem 2.3 implies that $U_{ad}(e) \neq \emptyset$ for every $e \in E_{ad}$.

Theorem 3.1 *There exists a solution of the Optimal control problem $\mathcal{P}$.*

Proof. Let $\{(u_n, e_n)\} \in U_{ad}(e_n) \times E_{ad}$ be a minimizing sequence i.e.

$$\lim_{n\to\infty} J(u_n, e_n) = \inf_{U_{ad}(e)\times E_{ad}} J(u, e).$$

There exists $(u_*, e_*) \in \mathcal{K}(e_*) \times E_{ad}$ and a subsequence denoted again by (u_n, e_n) such that

$$e_n \to e_* \text{ in } H^2(0,T),\ u_n \rightharpoonup u_* \text{ in } \mathcal{V}. \tag{22}$$

The elements $u_n \in U_{ad}(e_n)$ are weak solutions of the State problem (1)-(4) with $e \equiv e_n$ and fulfil

$$\langle\langle \ddot{u}_n, e_n y\rangle\rangle + \int_Q \left[d e_n^3(x) u_{n_{xx}} y_{xx} - f(t,x) y(t,x)\right] dt\, dx = \langle\langle g_n, y\rangle\rangle\, \forall\, y \in \mathcal{V} \tag{23}$$

with functionals $g_n \in \mathcal{V}^*,\ n \in \mathbb{N}$ fulfilling

$$\langle\langle g_n, v\rangle\rangle \geq 0\, \forall\, v \in V,\ v \geq 0;\ \langle\langle g_n, u_n - \Phi - \frac{1}{2} e_n\rangle\rangle = 0. \tag{24}$$

After inserting $y(x) \equiv 1$ in (23) we obtain

$$\int_0^L [e_n(x)(\dot{u}_n(T,x) - v_0(x))\, dx - \int_Q f(t,x)\, dt\, dx = \langle\langle g_n, 1\rangle\rangle.$$

Using the definition of the admissible set U_{ad} and the property (24) of the functionals g_n we arrive to the estimates

$$\|g_n\|_{\mathcal{V}^*} \leq c_5,\ \|\ddot{u}_n\|_{\mathcal{V}^*} \leq c_6.$$

Then there exist the subsequence of $\{u_n, e_n, g_n\}$ (denoted by $\{u_n, e_n, g_n\}$) fulfilling the convergence (22) and $g_n \rightharpoonup^* g$ in $\mathcal{V}^*$ such that $u_* \in U_{ad}(e_*)$ with a contact functional $g \equiv g_*$. Lower semicontinuity properties of the functional J imply

$$J(u_*, e_*) \leq \liminf_{n\to\infty} J(u_n, e_n) = \inf_{U_{ad}(e)\times E_{ad}} J(u, e).$$

Then

$$J(u_*, e_*) = \min_{U_{ad}(e)\times E_{ad}} J(u, e)$$

and the proof is complete.

Remark 3.2 *The family of convex sets $\{\mathcal{K}(e)\}$ fulfils the "continuity" condition*

$$e_n \to e_0 \textit{ in } H^2(\Omega) \Rightarrow \mathcal{K}(e_0) = \mathrm{Lim}_{n\to\infty} \mathcal{K}(e_n),\ e_j \in E_{ad},\ j = 0, 1, ..., n, ...,$$

where the symbol Lim *means the special type of convergence introduced by U.Mosco in [5], see also [3]. This property was used in the proof of the previous theorem.*

Remark 3.3 *We have chosen an admissible set $U_{ad}(e)$ in a form (21) because there are no uniqueness and no a priori estimates of solutions of the state variational inequality. The smoothness assumption $e \in H^3(0,L)$ is inevitable due to the appearance of the control parameter e in the term connected with the second derivative $\ddot{u} \in (L_\infty(I,V))^*$.*

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