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EXAMPLES OF NON-HYPERREFLEXIVE REFLEXIVE SPACES OF OPERATORS

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1. Introduction and Preliminaries

Let $\mathcal{H}, \mathcal{H}'$ be complex separable Hilbert spaces and $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ be the space of all bounded linear operators $\mathcal{H} \to \mathcal{H}'$. For $\mathcal{H} = \mathcal{H}'$ we shall write briefly $\mathcal{L}(\mathcal{H}, \mathcal{H}) = \mathcal{L}(\mathcal{H})$. The *reflexive closure* of $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is defined by

$$\operatorname{Ref} \mathcal{S} = \bigcap_{x \in \mathcal{H}} \{T \in \mathcal{L}(\mathcal{H}, \mathcal{H}'); \quad Tx \in [\mathcal{S}x]\}$$

where $[\mathcal{S}x]$ denotes the closed linear span of $\mathcal{S}x = \{Sx;\ S \in \mathcal{S}\}$. For $T \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, let

$$\mathrm{d}\,(T, \mathcal{S}) = \inf_{S \in \mathcal{S}} \|T - S\| = \inf_{S \in \mathcal{S}} \sup_{x \in \mathcal{H}, \|x\| \le 1} \|Tx - Sx\|$$

and

$$\alpha\,(T, \mathcal{S}) = \sup_{x \in \mathcal{H}, \|x\| \le 1} \inf_{S \in \mathcal{S}} \|Tx - Sx\|.$$

It is well-known [4, 8] that

(i) $\alpha\,(T, \mathcal{S}) \le \mathrm{d}\,(T, \mathcal{S})$ and
(ii) $\operatorname{Ref} \mathcal{S}$ is a WOT (weak operator topology) closed subspace of $\mathcal{L}(\mathcal{H}, \mathcal{H}')$.
(iii) $\alpha\,(T, \mathcal{S}) = \sup\{\|QTP\| : P, Q$ are orthogonal projections and $Q\mathcal{S}P = \{0\}\}$.
(iv) $\alpha\,(T, \mathcal{S}) = \sup\{|(Tx, y)| : \|x\| = \|y\| = 1,\ (Sx, y) = 0$ for all $S \in \mathcal{S}\}$

Definition 1.1. A WOT closed subspace $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is said to be *reflexive* if $\operatorname{Ref} \mathcal{S} = \mathcal{S}$ and it is called *hyperreflexive* if there exists a constant $c \ge 1$ such that

$$\mathrm{d}\,(T, \mathcal{S}) \le c\,\alpha\,(T, \mathcal{S}) \qquad \forall\, T \in \mathcal{L}(\mathcal{H}, \mathcal{H}')\,. \tag{1}$$

The number $\kappa(\mathcal{S}) = \inf\{c \ge 1\,;\ c$ is satisfying (1)$\}$ is called the *hyperreflexivity constant* of $\mathcal{S}$.

A single operator $T \in \mathcal{L}(\mathcal{H})$ is (hyper)reflexive if so is the unital weakly closed algebra generated by T.

The reflexivity of subalgebras was studied for the first time in 1966 by D. Sarason [9]. The notion of reflexivity of algebras of operators was generalized to subspaces of operators by V.S. Shulʹman [10]. The concept of hyperreflexivity for algebras was introduced in 1975, [1, 2] and generalized for subspaces in 1985 [6, 7].

It easy to see that every hyperreflexive linear space $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$ is reflexive. On the other hand, there are reflexive linear spaces of operators that are not hyperreflexive. The aim of this paper is to give a review of known examples of nonhyperreflexive reflexive spaces. In fact there is only a few such examples and we shall show that all of them can be viewed as modifications of the Kraus-Larson example [6].

First, let us recall the following results on hyperreflexivity of similar and unitary equivalent subspaces of operators [3].

Proposition 1.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Banach spaces and let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{X})$ be a hyperreflexive subspace of operators. If $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ are invertible operators, then $A\mathcal{S}B \subseteq \mathcal{L}(\mathcal{Y})$ is a hyperreflexive subspace and*

$$\tfrac{1}{\|A\|\|B\|\|A^{-1}\|\|B^{-1}\|}\kappa(\mathcal{S}) \le \kappa(A\mathcal{S}B) \le \|A\|\|B\|\|A^{-1}\|\|B^{-1}\|\kappa(\mathcal{S}).$$

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Corollary 1.3. *Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$ be a hyperreflexive linear space. If U and V are unitary operators on $\mathcal{H}$, then the space $U\mathcal{S}V$ is hyperreflexive and $\kappa(U\mathcal{S}V) = \kappa(\mathcal{S})$.*

All known examples of non-hyperreflexive reflexive spaces are direct sum of hyperreflexive subspaces. In their constructions the following result (see, e.g. [5]) is used:

Theorem 1.4. *For $n \in \mathbb{N}$ let $\mathcal{H}_n$ be a Hilbert space and let $\mathcal{S}_n \subset \mathcal{L}(\mathcal{H}_n)$ be a subspace. Then $\mathcal{S} = \bigoplus_{n=1}^{\infty} \mathcal{S}_n$ is hyperreflexive if and only if all $\mathcal{S}_n$ are hyperreflexive and there is $K > 0$ such that $\kappa(\mathcal{S}_n) \leq K$ for all $n \in \mathbb{N}$.*

2. Kraus-Larson example

Now we are going to describe the first known example of non-hyperreflexive reflexive space [6]. Let H_2 be a two-dimensional Hilbert space with orthonormal basis $\{e_1, e_2\}$. Fix $0 < \varepsilon < 1/3$ and put $u_1 = e_1$, $u_2 = e_1 + \varepsilon e_2$.

Lemma 2.5. *Let $0 < \varepsilon < 1/3$, and let*

$$\mathcal{S}_\varepsilon = \left\{ S_{\lambda,\mu} = \begin{pmatrix} 0 & \lambda \\ \mu & -(\lambda+\mu)/\varepsilon \end{pmatrix} : \ \lambda, \mu \in \mathbb{C} \right\} .$$

Then $\mathcal{S}_\varepsilon$ is a hyperreflexive subspace of $\mathcal{L}(H_2)$ with

$$\kappa(\mathcal{S}) \geq \frac{1}{3\varepsilon} . \tag{2}$$

Using Corollary 1.3 and the following theorem (2) can be improved.

Theorem 2.6 (S. Tosaka [12])**.** *Let $\mathcal{H} = \mathbb{C}^2$ and let $\mathcal{L}$, $\mathcal{M}$ be one-dimensional subspaces of $\mathcal{H}$ such that $\mathcal{L} + \mathcal{M} = \mathcal{H}$. Denote* $\mathrm{Alg}\{\mathcal{L}, \mathcal{M}\} = \{T \in B(\mathcal{H});\ T\mathcal{L} \subseteq \mathcal{L}$ *and* $T\mathcal{M} \subseteq \mathcal{M}\}$. *Thus,* $\mathrm{Alg}\{\mathcal{L}, \mathcal{M}\}$ *is the algebra of all operators in $B(\mathcal{H})$ that leave $\mathcal{L}$ and $\mathcal{M}$ invariant . If the angle φ between $\mathcal{L}$ and $\mathcal{M}$ is not zero, then* $\mathrm{Alg}\{\mathcal{L}, \mathcal{M}\}$ *is hyperreflexive and its hyperreflexivity constant is* $\kappa(\mathrm{Alg}\{\mathcal{L}, \mathcal{M}\}) = \frac{1}{\sin\varphi}$.

Lemma 2.7. *Under the notation of Lemma 2.5 the following estimation holds.*

$$\kappa(\mathcal{S}_\varepsilon) = \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} > \frac{1}{\varepsilon} . \tag{3}$$

Proof. Observe that $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is unitary and for $\forall\, \lambda, \mu \in \mathbb{C}$

$$U S_{\lambda,\mu} = U \begin{pmatrix} 0 & \lambda \\ \mu & -(\lambda+\mu)/\varepsilon \end{pmatrix} = \begin{pmatrix} -\mu & (\lambda+\mu)/\varepsilon \\ 0 & \lambda \end{pmatrix} .$$

This means that $U\mathcal{S}_\varepsilon = \mathrm{Alg}\{u_1, u_2\}$, $u_1 = e_1$, $u_2 = e_1 + \varepsilon e_2$, and by Theorem 2.6 and Corollary 1.3

$$\kappa(\mathcal{S}_\varepsilon) = \frac{1}{\sin\varphi} ,$$

where $\cos\varphi = \frac{(u_1, u_2)}{\|u_1\|\|u_2\|} = \frac{1}{\sqrt{1+\varepsilon^2}}$. Consequently $\sin\varphi = \sqrt{1 - \cos^2\varphi} = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$. From this (3) follows easily. ☐

Now putting $\mathcal{S}_n = \mathcal{S}_{1/n}$, $\mathcal{H}_n = \mathbb{C}^2$, $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ and $\mathcal{S} = \bigoplus_{n=1}^{\infty} \mathcal{S}_n$ we obtain the Kraus-Larson example [6] with slightly improved estimate $\kappa(\mathcal{S}_n) = n\sqrt{1 + \frac{1}{n^2}} > n$.

3. Non-hyperreflexive reflexive intertwiners

The Kraus-Larson technique can be used also to obtain non-hyperreflexive reflexive intertwiners and to show that quasi-similarity does not preserve hyperreflexivity [13]. Recall that the intertwiners of $T \in \mathcal{L}(\mathcal{H}), T' \in \mathcal{L}(\mathcal{H}')$ is defined by

$$I(T, T') = \{X \in \mathcal{L}(\mathcal{H}, \mathcal{H}') : \ XT = T'X\} .$$

The construction of non-hyperreflexive reflexive intertwiner in [13, Section 5] is based on the following observation: Putting

$$A_n = \begin{pmatrix} 0 & n \\ 0 & 1 \end{pmatrix} , \ B_n = \begin{pmatrix} 0 & 0 \\ -n & 1 \end{pmatrix}$$

we obtain

$$X \in I(A_n, B_n) \iff X_n = \begin{pmatrix} 0 & \lambda \\ \mu & -n(\lambda+\mu) \end{pmatrix},$$

i.e. $I(A_n, B_n) = \mathcal{S}_n$ from the Kraus-Larson example. Now it is easy to prove the following theorem

Theorem 3.8. *There exist operators T, T' for which $I(T, T')$ is reflexive but not hyperreflexive.*

Proof. It is enough to put

$$T_n = e^{\pi/n}\frac{1}{n}(nI + A_n), \qquad T'_n = e^{\pi/n}\frac{1}{n}(nI + B_n).$$

Since $\|A_n\| = \|B_n\| = \sqrt{1+n^2}$ we have $\|T_n\| \le 1 + \frac{\sqrt{1+n^2}}{n}$. Consequently $\{\|T_n\|\}$ is a bounded sequence. By analogous reasoning $\{\|T'_n\|\}$ is also bounded. For $n \neq m$ the minimal polynomials of T_n and T'_m are relatively prime. It follows that $I(T_n, T'_m) = \{0\}$ and If

$$T = \bigoplus_{n=1}^{\infty} T_n, \quad T' = \bigoplus_{n=1}^{\infty} T'_n, \quad \text{then } I(T, T') = \bigoplus_{n=1}^{\infty} I(T_n, T'_n).$$

Thus the Kraus-Larson example is also an example of intertwiner which is reflexive but not hyperreflexive. □

4. C_0 contractions

The construction from Theorem 3.8 can be further modified to obtain a contraction T of class C_0 (see [11] for the definition) having reflexive but not hyperreflexive commutant $\{T\}'$. Put again $A_n = \left(\begin{smallmatrix} 0 & n \\ 0 & 1 \end{smallmatrix}\right)$ and $D_n = (1 - \frac{1}{n})I + \frac{1}{n^2}A_n$, $T_n = \frac{e^{i\pi/n}}{\|D_n\|}D_n$ Using Theorem 2.6 it easy to prove that $\kappa\{T_n\}' = \kappa\{A_n\}' = \sqrt{1+n^2}$ [3, Lemma 1.9]. Thus we obtain a sequence of contraction $\{T_n\}_{n=1}^{\infty}$ having the following properties:

(i) $\|T_n\| = 1$.

(ii) $D_n \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \begin{pmatrix} 1/n \\ 1-(1/n)+(1/n^2) \end{pmatrix}$. Therefore the spectrum $\sigma(T_n) = \{\lambda_n, \mu_n\}$, $|\lambda_n| = \frac{1-(1/n)}{\|D_n\|} < |\mu_n| = \frac{1-(1/n)+(1/n^2)}{\|D_n\|} < 1$, $\lim|\lambda_n| = \lim|\mu_n| = 1$.

(iii) If $m \neq n$ then $\sigma(T_n) \cap \sigma(T_m) = \emptyset$.

Since $\lim(1 - |\lambda_n|) + (1 - |\mu_n|) = 0$ there exists a subsequence $\{T_k\}_{k=1}^{\infty}$ of $\{T_n\}_{n=1}^{\infty}$ such that the following theorem holds.

Theorem 4.9. *There exists a sequence of matrices $T_k \in C^{2\times 2}$ such that*

(1) *$\|T_k\| = 1$ for all $k = 1, 2, \ldots$.*

(2) *Each T_k has two eigenvalues $\lambda_k \neq \mu_k$ and therefore its commutant $\{T_k\}'$ is hyperreflexive.*

(3) *For any $k \neq m$ the spectra of T_k and T_m are disjoint, i.e. $\{\lambda_k, \mu_k\} \cap \{\lambda_m, \mu_m\} = \emptyset$.*

(4) $\lim\limits_{k\to\infty} \kappa(\{T_k\}') = \infty$.

(5) $\sum\limits_{k=1}^{\infty} [(1 - |\lambda_k|) + (1 - |\mu_k|)] < \infty$ *and, consequently,*

(6) *Blaschke product $B(\lambda) = \prod\limits_{k=1}^{\infty} \frac{\overline{\lambda_k}}{|\lambda_k|}\frac{\lambda_k - \lambda}{1 - \overline{\lambda_k}\lambda}\frac{\overline{\mu_k}}{|\mu_k|}\frac{\mu_k - \lambda}{1 - \overline{\mu_k}\lambda}$ converges in the open unit disk.*

Consequently, $T = \bigoplus_{k=1}^{\infty} T_k$ is a C_0 contraction having minimal function $B(\lambda)$ and $\{T\}'$ is reflexive but not hyperreflexive.

The above obtained C_0 contraction T is not a model operator, i.e there is no inner function θ such that $T = S_\theta$, where $S_\theta \in \mathcal{L}(H^2 \ominus \theta H^2)$, $S_\theta u = P_\theta[\lambda u(\lambda)]$. Here H^2 and H^∞ are the usual Hardy spaces of functions analytic in the unit disk, θ is inner if $|\theta(e^{it})| = 1$ almost everywhere and P_θ denotes the orthogonal projection from H^2 onto $\mathcal{H}_\theta = H^2 \ominus \theta H^2$.

Recently [3] a Blaschke product $B(\lambda)$ was constructed such that S_B is reflexive, but not hyperreflexive. First, the following sufficient condition was proved.

Theorem 4.10. *For $\lambda \in \mathbb{C}$, $|\lambda| < 1$ denote the corresponding Blaschke factor by*

$$b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{\lambda - z}{1 - \bar{\lambda} z}$$

and for a Blaschke product having only simple zeroes

$$B(z) = \prod_{n=1}^{\infty} b_{\lambda_n}(z) \text{ put } B_{\lambda_n}(z) = \frac{B(z)}{b_{\lambda_n}(z)} .$$

If B satisfies the Carleson condition

$$\inf_n |B_{\lambda_n}(\lambda_n)| > 0 ,$$

then S_B is hyperreflexive.

The main idea (due to R.V. Bessonov) which allows to construct a Blaschke product B having simple zeroes for which S_B is not hyperreflexive was to take $B(z) = C(z)D(z)$, where $C(z) = \prod_{n=1}^{\infty} b_{\mu_n}(z)$, $D(z) = \prod_{n=1}^{\infty} b_{\nu_n}(z)$ such that $|\mu_n - \nu_n|$ is sufficiently small, i.e. B is 'almost' a square. Then it can be shown that S_B is similar to direct sum of its restrictions M_n to the 2-dimensional spaces spanned by the eigenvectors corresponding to the eigenvalues μ_n and ν_n of the model operator S_B. So this example is again of the Kraus-Larson type.

We conclude with a natural open problem:

Question 4.11. Does there exist a non-hyperreflexive reflexive space of operators which is not similar to a direct sum of reflexive spaces?

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