ON HYPERREFLEXIVITY OF POWER PARTIAL ISOMETRIES

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Let $\mathcal{H}$ be a complex separable Hilbert space. Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For an operator $T \in B(\mathcal{H})$ let us consider $\mathcal{W}(T)$ a unital subalgebra of $B(\mathcal{H})$ containning the operator T and closed in WOT topology. Denote by $\operatorname{Lat} T$ the set of all projections onto closed subspaces invariant for operator T. Now for a given operator $A \in B(\mathcal{H})$ except the usual distance from A to $\mathcal{W}(T)$ denoted by $\operatorname{dist}(A, \mathcal{W}(T))$, we can define the distance ,,determined by its invariant subspaces" as $\alpha(A, \mathcal{W}(T)) = \sup\{\|(I-P)AP\| : P \in \operatorname{Lat} T\}$. Usually $\alpha(A, \mathcal{W}(T)) \leqslant \operatorname{dist}(A, \mathcal{W}(T))$. The operator $T \in B(\mathcal{H})$ is called *hyperreflexive* if the usual distance can be controlled by the distance α, i.e. there is a positive constant κ such that

$$\operatorname{dist}(A, \mathcal{W}(T)) \leqslant \kappa\, \alpha(A, \mathcal{W}(T)) \quad \text{for all} \quad A \in B(\mathcal{H}).$$

Recall that an operator $V \in B(\mathcal{H})$ is called a *partial isometry* if V^*V is an orthogonal projection. An operator S is a *power partial isometry* if S^n is a partial isometry for every positive integer n. It is known (see [4]) that if S is a power partial isometry on $\mathcal{H}$ then there is a unique orthogonal decomposition $\mathcal{H} = \mathcal{H}_u(S) \oplus \mathcal{H}_s(S) \oplus \mathcal{H}_c(S) \oplus \mathcal{H}_t(S)$ where $\mathcal{H}_u(S)$, $\mathcal{H}_s(S)$, $\mathcal{H}_c(S)$, $\mathcal{H}_t(S)$ reduce S and $S_u = S|_{\mathcal{H}_u(S)}$ is a unitary operator, $S_s = S|_{\mathcal{H}_s(S)}$ is a unilateral shift of arbitrary multiplicity, $S_c = S|_{\mathcal{H}_c(S)}$ is a backward shift of arbitrary multiplicity and $S_t = S|_{\mathcal{H}_t(S)}$ is (possibly infinite) direct sum of truncated shifts.

Reflexivity (the weaker property then hyperreflexivity) of power partial isometries was studied in [1]. It is known that the unilateral shift is hyperreflexive [2]. A backward shift is also hyperreflexive since hyperreflexivity is preserved after taking the adjoint of the operator. On the other hand the single Jordan block is not hyperreflexive not even reflexive [3]. Conditions for hyperreflexivity of power partial isometries will be presented.

For a power partial isometry S let us define decreasing sequences of projections $P_n = S^{*n}S^n$, $Q_n = S^nS^{*n}$ for all positive integers n. (We are setting the convention that $S^0 = I$.) Denote $\overline{d}_k = \dim \mathcal{R}(P_{k-1}(Q_0 - Q_1))$, $d_k = \dim \mathcal{R}(P_{k-1}(Q_0-Q_1)) \ominus \mathcal{R}(P_k(Q_0-Q_1))$ for $k \in \mathbb{N}$. Denote also $\overline{d}_\infty = d_\infty = \dim \bigcap_{k\in\mathbb{N}} \mathcal{R}(P_{k-1}(Q_0 - Q_1))$. Let us observe that the number $\overline{d}_k$ says how many forward shifts (truncated or not) of order at least k are in operator S, the number d_k says how many forward shifts (truncated or not) of order exactly k are in operator S. Symmetrically we denote $\overline{d}^*_k = \dim \mathcal{R}(Q_{k-1}(P_0 - P_1))$, $d^*_k = \dim \mathcal{R}(Q_{k-1}(P_0-P_1)) \ominus \mathcal{R}(Q_k(P_0-P_1))$ for $k \in \mathbb{N}$ and $\overline{d}^*_\infty = d^*_\infty = \dim \bigcap_{k\in\mathbb{N}} \mathcal{R}(Q_{k-1}(P_0 - P_1))$.

Theorem. *Let $S \in B(\mathcal{H})$ be a completely non–unitary power partial isometry. If*

(i) $d_\infty > 0$ *or*

(ii) $d^*_\infty > 0$ *or*

(iii) *there is $k_0 \in \mathbb{N}$ such that $d_k = 0$ for $k > k_0$ and $d_{k_0} + d_{k_0-1} \geqslant 2$*

then S is hyperreflexive.

References

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