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REFLEXIVITY AND HYPERREFLEXIVITY FOR SETS OF OPERATORS

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1. Introduction

In [5, 3] Shulman and Loginov extended the notion of a reflexive algebra of operators to the context of linear spaces of operators as follows. Let $\mathscr{X}$ be a complex Banach space and let $\mathcal{B}(\mathscr{X})$ be the Banach algebra of all bounded linear operators on $\mathscr{X}$. For a linear subspace $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$, one defines its reflexive cover by

$$\mathrm{Ref}(\mathcal{S}) = \big\{T \in \mathcal{B}(\mathscr{X}); \quad Tx \in [\mathcal{S}x], \quad \text{for all } x \in \mathscr{X}\big\},$$

where $[\mathcal{S}x]$ denotes the closed linear span of $\mathcal{S}x = \{Sx;\ S \in \mathcal{S}\}$, the orbit of $\mathcal{S}$ at x. It is not hard to see that $\mathrm{Ref}(\mathcal{S})$ is a weakly closed linear space of operators which contains $\mathcal{S}$. If $\mathrm{Ref}(\mathcal{S}) = \mathcal{S}$, then $\mathcal{S}$ is said to be a reflexive space of operators.

In the above definition the linearity is not needed. Reflexive cover may be defined for an arbitrary nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ by

$$\mathrm{Ref}(\mathcal{M}) = \big\{T \in B(\mathscr{X}); \quad Tx \in \overline{\mathcal{M}x}, \quad \text{for all } x \in \mathscr{X}\big\}, \tag{1}$$

where $\overline{\mathcal{M}x}$ is just the closure of the orbit $\mathcal{M}x$ in $\mathscr{X}$. Of course, if $\mathcal{M}$ is a linear space, then $\overline{\mathcal{M}x} = [\mathcal{M}x]$, for all $x \in \mathscr{X}$. However, $\overline{\mathcal{M}x} \subset [\mathcal{M}x]$, in general.

It is obvious that $\mathcal{M} \subseteq \mathrm{Ref}(\mathcal{M})$ holds for every nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$. One can also see that $\mathrm{Ref}(\mathscr{M})$ is a strongly closed set. Thus, the following definition makes sense.

Definition 1.1. A strongly closed nonempty set $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ is reflexive if $\mathrm{Ref}(\mathcal{M}) = \mathcal{M}$.

Example 1.2. Let $\mathcal{M} = \{M_1, \dots, M_n\}$ be a finite set of operators. Then $\overline{\mathcal{M}x} = \mathcal{M}x$, for every $x \in \mathscr{X}$. Thus, if $T \in \mathrm{Ref}(\mathcal{M})$, then for each $x \in \mathscr{X}$ there exists $M_i \in \mathcal{M}$, which depends on x, such that $Tx = M_i x$, i.e., $x \in \ker(T - M_i)$. It follows that $\mathscr{X} = \bigcup_{i=1}^{n} \ker(T - M_i)$. Of course, $\mathscr{X}$ cannot be a finite union of proper subspaces. Therefore, $T = M_i$ for some $M_i \in \mathcal{M}$, which means that $\mathcal{M}$ is reflexive.

If a nonempty set $\mathcal{M}$ has some structure, then very often $\mathrm{Ref}(\mathcal{M})$ has a similar structure, as well. Without a proof we list a few examples in the following proposition.

Proposition 1.3. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be a nonempty set.*
(a) If $\mathcal{M}$ is convex, then $\mathrm{Ref}(\mathcal{M})$ *is convex.*
(b) If $\mathcal{M}$ is additive, then $\mathrm{Ref}(\mathcal{M})$ *is additive.*
(c) If $\mathcal{L} \subseteq \mathcal{B}(\mathscr{X})$ is such that $\mathcal{L}\mathcal{M} \subseteq \mathcal{M}$, then $\mathcal{L}\mathrm{Ref}(\mathcal{M}) \subseteq \mathrm{Ref}(\mathcal{M})$.

Note that it follows by Proposition 1.3 that $\mathrm{Ref}(\mathcal{M})$ is a complex (real) subspace in $\mathcal{B}(\mathscr{X})$ if $\mathscr{M}$ has this property. Also, if $\mathscr{M}$ is a multiplicative semigroup (algebra), then $\mathrm{Ref}(\mathcal{M})$ is a multiplicative semigroup (algebra).

If $\mathcal{S} \subseteq \mathcal{B}(\mathscr{X})$ is a closed linear space, then

$$\mathrm{d}(T, \mathcal{S}) = \inf_{S \in \mathcal{S}} \|T - S\| = \inf_{S \in \mathcal{S}} \sup_{\substack{x \in \mathscr{X} \\ \|x\|=1}} \|Tx - Sx\| \qquad \big(T \in \mathcal{B}(\mathscr{X})\big) \tag{2}$$

defines a seminorm on $\mathcal{B}(\mathscr{X})$, i.e., by

$$\|T+\mathcal{S}\| = \mathrm{d}(T,\mathcal{S}) \qquad \big(T\in\mathcal{B}(\mathscr{X})\big)$$

is given a norm on the quotient space $\mathcal{B}(\mathscr{X})/\mathcal{S}$. Another seminorm on $\mathcal{B}(\mathscr{X})$ is the Arveson distance which is defined by

$$\alpha(T,\mathcal{S}) = \sup_{\substack{x\in\mathscr{X}\\ \|x\|=1}} \inf_{S\in\mathcal{S}} \|Tx-Sx\| \qquad \big(T\in\mathcal{B}(\mathscr{X})\big). \tag{3}$$

One has $\alpha(T,\mathcal{S})\leq \mathrm{d}(T,\mathcal{S})$, for every $T\in\mathcal{B}(\mathscr{X})$. On the other hand, for some subspaces $\mathcal{S}\subseteq\mathcal{B}(\mathscr{X})$ there exists a constant $c\geq 1$ such that

$$\mathrm{d}(T,\mathcal{S}) \leq c\alpha(T,\mathcal{S}) \qquad \big(T\in\mathcal{B}(\mathscr{X})\big). \tag{4}$$

A space satisfying this condition is said to be hyperreflexive. The concept was introduced by Arveson [1]. Note that every hyperreflexive space of operators is reflexive.

If one considers nonempty sets of operators instead of linear spaces, quantities defined in (2) and (3) still have sense although they are no more seminorms. Thus, for a nonempty set $\mathcal{M}\subseteq\mathcal{B}(\mathscr{X})$ we define

$$\mathrm{d}(T,\mathcal{M}) = \inf_{M\in\mathcal{M}} \sup_{\substack{x\in\mathscr{X}\\ \|x\|=1}} \|Tx-Mx\| \quad \text{and} \quad \alpha(T,\mathcal{M}) = \sup_{\substack{x\in\mathscr{X}\\ \|x\|=1}} \inf_{M\in\mathcal{M}} \|Tx-Mx\| \quad \big(T\in\mathcal{B}(\mathscr{X})\big). \tag{5}$$

Similarly as in the case of linear spaces one has $\alpha(T,\mathcal{M})\leq \mathrm{d}(T,\mathcal{M})$ for every $T\in\mathcal{B}(\mathscr{X})$. Also,

$$\mathrm{d}(T,\mathcal{M})=0 \iff T\in\mathcal{M} \quad \text{and} \quad \alpha(T,\mathcal{M})=0 \iff T\in\mathrm{Ref}(\mathcal{M}). \tag{6}$$

Definition 1.4. A nonempty set $\mathcal{M}\subseteq\mathcal{B}(\mathscr{X})$ is hyperreflexive if

$$\mathrm{d}(T,\mathcal{M}) \leq c\alpha(T,\mathcal{M}) \qquad \big(T\in\mathcal{B}(\mathscr{X})\big) \tag{7}$$

holds for some constant $c\geq 1$. The minimal constant c such that (7) holds is the hyperreflexivity constant for $\mathcal{M}$ and is denoted by $\kappa(\mathcal{M})$.

It follows from (6) that every hyperreflexive set of operators is reflexive as well.

As a simple example of a hyperreflexive set of operators we mention the closed unit ball $\mathcal{K}$ in $\mathcal{B}(\mathscr{X})$. It is not hard to see that

$$\mathrm{d}(T,\mathcal{K}) = \alpha(T,\mathcal{K}) = \|T\|-1 \qquad \big(T\in\mathcal{B}(\mathscr{X}),\ \|T\|>1\big),$$

which gives $\kappa(\mathcal{K})=1$.

2. Sets of operators defined by numerical range

For a complex Banach space $\mathscr{X}$, let $\mathscr{X}^*$ be the topological dual of $\mathscr{X}$. Denote

$$\Pi(\mathscr{X}) = \big\{(x,\xi)\in\mathscr{X}\times\mathscr{X}^*;\quad \langle x,\xi\rangle = 1 = \|x\| = \|\xi\|\big\}.$$

By Hahn-Banach Theorem, for every $x\in\mathscr{X}$ there exists $\xi\in\mathscr{X}^*$ such that $\|\xi\|=1$ and $\langle x,\xi\rangle=\|x\|$. In particular, we see that for every $x\in\mathscr{X}$ of norm one there is at least one pair in $\Pi(\mathscr{X})$ which has x as the firts entry. Note that in the case of a Hilbert space $\mathscr{H}$ one has $\Pi(\mathscr{H})=\{(x,x);\quad x\in\mathscr{H},\ \|x\|=1\}$.

Numerical range of an operator $A\in\mathcal{B}(\mathscr{X})$ is defined by

$$W(A) = \big\{\langle Ax,\xi\rangle;\quad (x,\xi)\in\Pi(\mathscr{X})\big\}.$$

It is well known that $W(A)$ is a convex subset of complex plane (Toeplitz-Hausdorff Theorem) such that $\sigma(A)\subseteq\overline{W(A)}\subseteq\{z\in\mathbb{C};\ |z|\leq\|A\|\}$.

For a subset $K \subseteq \mathbb{C}$, let

$$\mathcal{M}_K = \{M \in \mathcal{B}(\mathscr{X}); \quad W(M) \subseteq K\}. \tag{8}$$

The following proposition lists some obvious observations.

Proposition 2.1. *Let $K \subseteq \mathbb{C}$. Then*

(a)
- $\mathcal{M}_K = \emptyset \iff K = \emptyset$,
- $\mathcal{M}_K = \{0\} \iff K = \{0\}$,
- $\mathcal{M}_K = \mathcal{B}(\mathscr{X}) \iff K = \mathbb{C}$;

(b) *if K is closed, then $\mathcal{M}_K$ is weakly closed;*
(c) *if K is convex, then $\mathcal{M}_K$ is convex;*
(d) *if K is additive, then $\mathcal{M}_K$ is additive; and*
(e) *if $\Lambda \subseteq \mathbb{C}$ is such that $\Lambda K \subseteq K$, then $\Lambda \mathcal{M}_K \subseteq \mathcal{M}_K$ as well.*

Note that, by (e) in the previous proposition, $\mathcal{M}_K$ is a real subspace in $\mathcal{B}(\mathscr{X})$ if $\mathbb{R}K \subseteq K$. As a special set of this type we mention hermitian operators on $\mathscr{X}$, i.e.,

$$\mathcal{H}_{\mathscr{X}} = \{H \in \mathcal{B}(\mathscr{X}); \quad W(H) \subseteq \mathbb{R}\}.$$

Thus, $\mathcal{H}_{\mathscr{X}}$ is a real, but not a complex, subspace in $\mathcal{B}(\mathscr{X})$.

Proposition 2.2. *If $K \subseteq \mathbb{C}$ is a nonempty closed set, then $\mathcal{M}_K$, which is given by* (8)*, is a reflexive set of operators.*

Proof. Let $T \in \mathrm{Ref}(\mathcal{M}_K)$ be arbitrary. For every pair $(x, \xi) \in \Pi(\mathscr{X})$, we have to see that $\langle Tx, \xi\rangle \in K$. Thus, let $(x, \xi) \in \Pi(\mathscr{X})$ be arbitrary and let $\varepsilon > 0$. Then there exists $M_{x,\varepsilon} \in \mathcal{M}_K$ such that $\|Tx - M_{x,\varepsilon}x\| < \varepsilon$. It follows $|\langle Tx, \xi\rangle - \langle M_{x,\varepsilon}x, \xi\rangle| \leq \|Tx - M_{x,\varepsilon}x\| < \varepsilon$. As $\varepsilon > 0$ is arbitrary and $\langle M_{x,\varepsilon}x, \xi\rangle \in K$ we conclude that $\langle Tx, \xi\rangle \in K$. □

In particular, the set of hermitian operators in $\mathcal{B}(\mathscr{X})$ is reflexive.

In the case of a Hilbert space $\mathscr{H}$, the hermitian operators are precisely selfadjoint operators in $\mathcal{B}(\mathscr{H})$ and therefore, as it is well known, $\mathcal{H}_{\mathscr{H}} + i\mathcal{H}_{\mathscr{H}} = \mathcal{B}(\mathscr{H})$. Thus, an arbitrary operator $T \in \mathcal{B}(\mathscr{H})$ has a unique representation $T = H + iK$, where $H,\ K \in \mathcal{H}_{\mathscr{H}}$. It can be seen that

$$\mathrm{d}(T, \mathcal{H}_{\mathscr{H}}) = \alpha(T, \mathcal{H}_{\mathscr{H}}) = \|K\|,$$

which says that $\mathcal{H}_{\mathscr{H}}$ is a hyperreflexive set of operators with the hyperreflexivity constant 1. Actually the same holds for every complex Banach space $\mathscr{X}$ satisfying condition $\mathcal{H}_{\mathscr{X}} + i\mathcal{H}_{\mathscr{X}} = \mathcal{B}(\mathscr{X})$. However, there are Banach spaces such that the set of hermitian operators is quite small, even trivial.

Example 2.3. Let $\mathscr{X} = \ell^p$, where $1 \leq p \leq \infty$ and $p \neq 2$. Then the set of hermitian operators contains precisely operators of the form

$$(x_1, x_2, \ldots) \mapsto (a_1x_1, a_2x_2, \ldots) \qquad \big((x_1, x_2, \ldots) \in \ell^p\big),$$

where $(a_1, a_2, \ldots)$ is a bounded real sequence, i.e., hermitians are precisely those which have a real diagonal matrix with respect to the standard basis in ℓ^p (see [2] for the details). Thus, in this case, $\mathcal{H}_{\mathscr{X}} + i\mathcal{H}_{\mathscr{X}}$ are those operators with a diagonal matrix, which means, of course, that $\mathcal{H}_{\mathscr{X}} + i\mathcal{H}_{\mathscr{X}}$ is much smaller than $\mathcal{B}(\mathscr{X})$.

Example 2.4. Let $\mathscr{X} = \mathscr{C}^{(1)}[0, 1]$, the space of all continuously differentiable complex-valued functions on $[0, 1]$ with norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. For this space, $\mathcal{H}_{\mathscr{X}}$ is trivial, i.e., contains only the real scalar multiples of the identity operator (see [2]), which means that $\mathcal{H}_{\mathscr{X}} + i\mathcal{H}_{\mathscr{X}} = \mathbb{C}I$.

The following lemma, which we mention without a proof, is a helpful tool for showing that a set of operators is hyperreflexive.

Lemma 2.5. *Let $\mathcal{L} \subseteq \mathcal{M} \subseteq \mathcal{B}(\mathscr{X})$ be nonempty sets. Assume that $\mathcal{M}$ is hyperreflexive with hyperreflexivity constant $\kappa(\mathcal{M})$. If there exists a positive constant k such that*

$$\mathrm{d}(M, \mathcal{L}) \leq k\alpha(M, \mathcal{L}) \qquad \text{for all} \quad M \in \mathcal{M},$$

then $\mathcal{L}$ is a hyperreflexive set of operators and $\kappa(\mathcal{L}) \leq k + \kappa(\mathcal{M}) + k\kappa(\mathcal{M})$.

For instance, if $\mathscr{X} = \mathscr{C}^{(1)}[0,1]$, then we may conclude that the set of hermitian operators $\mathcal{H}_{\mathscr{X}} = \mathbb{R}I$ is hyperreflexive. Indeed, the complex space $\mathcal{H}_{\mathscr{X}} + i\mathcal{H}_{\mathscr{X}} = \mathbb{C}I$ is one-dimensional and therefore reflexive. By [4], every finite dimensional reflexive space of operators is hyperreflexive as well. Thus, by Lemma 2.5, $\mathcal{H}_{\mathscr{X}} = \mathbb{R}I$ is hyperreflexive.

For the space of hermitian operators in the case of Banach space from Example 2.3 we do not know if it is hyperreflexive because it is not known if the space of all diagonal operators on ℓ^p $(1 \leq p \leq \infty,\ p \neq 2)$ is hyperreflexive.

References

[1] W. Arveson, *Ten lectures on operator theory*, CBMS Regional Conference Series 55, AMS, 1984.

[2] E. Berkson, A. Sourour, *The hermitian operators on some Banach spaces*, Studia Math. **52** (1974), 33–41.

[3] A. I. Loginov, V. S. Šul'man, *Hereditary and intermediate reflexivity of W^*-algebras*, Math. USSR Izvestija **9**(6) (1975), 1189–1201.

[4] V. Müller, M. Ptak, *Hyperreflexivity of finite-dimensional subspaces*, J. Funct. Anal. **218** (2005), 395–408.

[5] V. S. Šul'man, *On reflexive operator algebras*, Math. USSR Sbornik **16**(2) (1972), 181–189.

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