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ON MULTIPLE SOLUTIONS OF GENERALIZED SECOND ORDER BOUNDARY VALUE PROBLEM WITH Φ-LAPLACIAN.

Boris Rudolf

We deal with the boundary value problem for a second order differential equation with Φ–Laplacian

$$(\Phi(x'))' = f(t, x, \Phi(x')) \tag{1}$$

with generalized boundary conditions

$$x'(0) = 0, \qquad x(b) = \int_0^b x(s)dg(s) - k\Phi(x'(b)). \tag{2}$$

We assume that $\Phi \in C^1(R)$ is an increasing function and $\lim_{x\to\pm\infty} \Phi(x) = \pm\infty$. Function $f : I \times R^2 \to R$ is a continuous function, $I = [0, b]$. Function g is a nondecreasing function of bounded variation, $k \geq 0$.

The aim of this paper is to prove the existence of multiple classical solutions $x(t) \in D$, $D = \{x \in C^1(I),\ \Phi(x') \in C^1(I)\}$ using a method of lower and upper solutions. The paper is motivated by the results of [2], [3], [5], [6].

Set $I^0 = I \setminus \{t_i;\ 0 < t_i < b,\ i = 1 \dots n\}$, $D^0 = \{x \in C(I) \cap C^1(I^0),\ \Phi(x') \in C^1(I)\}$.

Definition 1. A function $\alpha \in D^0$ is called a lower solution of (1), (2) if

$$\lim_{t\to t_i-} \alpha'(t) \leq \lim_{t\to t_i+} \alpha'(t) \qquad \text{for i=1, \dots ,n,}$$
$$(\Phi(\alpha'(t)))' \geq f(t, \alpha(t), \Phi(\alpha'(t))) \qquad \text{for } t \in I^0,$$
$$\alpha'(0) \geq 0, \qquad \alpha(b) \leq \int_0^b \alpha(s)dg(s) - k\Phi(\alpha'(b)).$$

Similarly a function $\beta \in D^0$ is called an upper solution of (1), (2) if

$$\lim_{t\to t_i-} \beta'(t) \geq \lim_{t\to t_i+} \beta'(t) \qquad \text{for i=1, \dots ,n,}$$
$$(\Phi(\beta'(t)))' \leq f(t, \beta(t), \Phi(\beta')), \qquad \text{for } t \in I^0,$$
$$\beta'(0) \leq 0, \qquad \beta(b) \geq \int_0^b \beta(s)dg(s) - k\Phi(\beta'(b)).$$

In the case of strict inequalities for limits at t_i, for the equation on I^0 and for the second boundary condition, we say that lower and upper solutions are strict.

Lemma 1. *[6] Let α, β be a strict lower and upper solution and $x(t)$ be a solution of the problem (1), (2).*

Then $\alpha(t) \leq x(t)$ implies $\alpha(t) < x(t)$ and $\beta(t) \geq x(t)$ implies $\beta(t) > x(t)$.

The following Lemma formulates the sufficient growth condition for the nonlinearity f. (Compare with [1], [3]).

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Lemma 2. *Let for each $r > r_0$ there exists a constant $a_r > 0$ and a function $h_r \in C(R_0^+, [a_r, \infty])$ satisfying*

$$\int_0^\infty \frac{\Phi^{-1}(s)}{h_r(s)}\, ds = \infty, \qquad \int_{-\infty}^0 \frac{\Phi^{-1}(s)}{h_r(|s|)}\, ds = -\infty \tag{3}$$

such that

$$|f(t, x, y)| < h_r(|y|) \tag{4}$$

for $t \in I,\ |x| < r,\ y \in R$.

Then for each $r > r_0$ there exists $\rho_r > 0$ such that for a solution x of (1), (2) $|x| < r$ implies $|x'| < \rho_r$.

Proof. Let $x_0 > 0$ be a constant such that $\Phi(x) > 0$ for $x > x_0$. Let $|x(t)| < r$ be a solution of (1), (2). Suppose that $x'(\tau) > x_0$ on (t_0, t). Substitute $\Phi(x'(t)) = y(t)$. Then

$$y'(t) = f(t, x, y) \le h_r(|y|)$$

and

$$\int_{t_0}^t \frac{\Phi^{-1}(y)y'}{h_r(|y|)}\, d\tau \le \int_{t_0}^t x'\, d\tau \le 2r.$$

Substitution $y(t) = s$ leads to

$$\int_{y(t_0)}^{y(t)} \frac{\Phi^{-1}(s)}{h_r(|s|)}\, ds \le 2r.$$

Then there exists $\rho_r > 0$ such that $y(t) < \Phi_{-1}(\rho_r)$. The proof is similar for the case $x'(\tau) < 0$.

Existence

Following existence theorems describe the situation for well ordered lower and upper solutions as well as for the unordered pair of lower and upper solutions.

Theorem 1. *Let $r > 0$ be such that*

(i) $f(t, r, 0) > 0$ and $f(t, -r, 0) < 0$ on I,

(ii) there exists a function $h_r \in C(R_0, [a_r, \infty])$ with $a_r > 0$ satisfying (3) such that (4) holds for $t \in I,\ |x| < r,\ y \in R$,

(iii) $G(b) < 1$.

Then there exists a solution x of (1), (2) such that $|x(t)| < r$.

Proof. Set $X = C^1([0, b])$ and $F_x(s) = \int_0^s f(\tau, x(\tau), \Phi(x'(\tau)))\, d\tau$.

We define an operator $T : X \to X$ by

$$Tx(t) = \frac{1}{G(b) - 1}\left\{\int_0^b G(s)\Phi^{-1}(F_x(s))\, ds + k(F_x(b))\right\} - \int_t^b \Phi^{-1}(F_x(s))\, ds.$$

Then $Tx(t) \in D = \{x \in C^1(I),\ \Phi(x') \in C^1(I)\}$.

Obviously $(Tx)'(0) = 0$. Also

$$Tx(b) = \int_0^b Tx(s)dg(s) - k\Phi(Tx'(b))$$

is fulfilled.

The operator $T : X \to X$ is completely continuous. A fixed point of T is a solution of (1), (2).

A perturbed boundary value problem

$$(5) \qquad (\Phi(x'))' = \lambda f(t,x,\Phi(x')) + (1-\lambda)x(t),$$

$$(2) \qquad x'(0) = 0, \qquad x(b) = \int_0^b x(s)dg(s) - k\Phi(x'(b)),$$

posses a strict lower solution $-r$ and a strict upper solution r for each $\lambda \in [0,1]$.

The homotopy operator

$$H(x,\lambda) = \frac{1}{G(b)-1}\left\{\int_0^b G(s)\Phi^{-1}(F_{x,\lambda}(s))\,ds + k(F_{x,\lambda}(b))\right\} - \int_t^b \Phi^{-1}(F_{x,\lambda}(s))\,ds,$$

with $F_{x,\lambda}(s) = \int_0^s \lambda f(\tau, x(\tau), \Phi(x'(\tau))) + (1-\lambda)x(\tau)\,d\tau$ is completely continuous.

Set $\Omega_{r,\varrho_r} = \{x \in X;\ |x| < r,\ |x'| < \varrho_r\}$. As a fixed point of H is a solution of (5), (2), Lemma 2 and Lemma 3 imply that there is no solution on the boundary of Ω_{r,ϱ_r}. Then the Leray-Schauder degree

$$d(H(.,\lambda), \Omega_{r,\varrho_r}, 0)$$

is well defined and independent on λ.

For $\lambda = 0$ is $H(x,0)$ an odd operator. Then

$$(6) \qquad d(I-T, \Omega_{r,\varrho_r}, 0) = d(I - H(x,0), \Omega_{r,\varrho_r}, 0) = 1 \pmod 2$$

which implies the existence of a fixed point $x \in \Omega_{r,\varrho_r}$ of T.

Theorem 2. *Let*

(i) $\alpha \le \beta$, $\alpha(t)$, $\beta(t)$ be a lower and upper solution of (1), (2),

(ii) $\exists h \in C(R_0^+, [a,\infty])$ with $a > 0$ satisfying (3) such that (4) holds for $t \in I$, $\alpha(t) \le x \le \beta(t)$, $y \in R$,

(iii) $G(b) < 1$.

Then there exists a solution x of (1),(2) such that $\alpha(t) \le x(t) \le \beta(t)$.

Proof. Set $r = \max\{||\alpha||,\ ||\beta||\}$. For $M > \max\{|f(t,x,y)|;\ t \in I,\ \alpha(t) \le x \le \beta(t),\ |y| < \varrho_r\}$ we consider a perturbation

$$(7) \qquad (\Phi(x'))' = f^*(t,x,\Phi(x')),$$

of the equation (1) where

$$f^*(t,x,y) = \begin{cases} f(t,\beta(t),y) + M(r-\beta(t)) + M & x > r+1, \\ f(t,\beta(t),y) + M(x-\beta(t)) & \beta(t) < x \le r+1, \\ f(t,x,y) & \alpha(t) \le x \le \beta(t), \\ f(t,\alpha(t),y) - M(\alpha(t)-x) & -r-1 \le x < \alpha(t), \\ f(t,\alpha(t),y) - M - M(\alpha(t)+r) & x < -r-1. \end{cases}$$

Then for each $\varepsilon > 0$

$$(\Phi(\alpha'(t) - \varepsilon)' > f^*(t, \alpha(t) - \varepsilon, \Phi(\alpha'(t) - \varepsilon).$$

That means $\alpha(t) - \varepsilon$ is a strict lower solution of the BVP (7),(2). Similarly $\beta(t) + \varepsilon$ is a strict upper solution of (7), (2).

Moreover $-(r+1)$, $r+1$ are also strict lower and upper solutions of (7), (2) and f^* satisfies (ii) of Theorem 2 with $h_{r+1}(s) = h(s) + (2r+1)M$.

Theorem 2 implies the existence of a solution x of (7), (2) satisfying $|x(t)| < r+1$.

We prove that $x(t) \ge \alpha(t)$. Assuming the contrary we suppose that $\max(\alpha(t) - x(t)) = \varepsilon > 0$. But $\alpha(t) - \varepsilon$ is a strict lower solution which is in a contradiction with $\alpha(t_0) - \varepsilon = x(t_0)$ due to Lemma 2. Then $\alpha(t) \le x(t)$. Similarly $x(t) \le \beta(t)$. That means $f^*(t,x,\Phi(x')) = f(t,x,\Phi(x'))$ and $x(t)$ is also a solution of (1), (2).

Moreover (6) holds on the set $\Omega = \{x \in X;\ \alpha < x < \beta,\ |x'| < \varrho_{r+1}\}$

Theorem 3. *Let*

(i) $\alpha \nleq \beta$, $\alpha(t)$, $\beta(t)$ *be strict lower and upper solutions of the problem (1), (2),*

(ii) for each $r > 0$ $\exists M_r > 0$ *such that* $|f(t,x,y)| \leq M_r$ *for each* $t \in I$, $|x| < r$, $y \in R$,

(iii) $G(b) < 1$.

Then there is a solution x *of (1), (2), such that* $\exists t_a \in I,\ \alpha(t_a) > x(t_a),\ \exists t_b \in I,\ x(t_b) > \beta(t_b)\}$.

Proof. Set $r_0 = \max(||\alpha||, ||\beta||)$, $r > r_0 + b\Phi^{-1}(2M_r b)$.

We define a perturbation f^* by

$$f^*(t,x,y) = \begin{cases} f(t,r,y) + M_r & x > r+1, \\ f(t,r,y) + M_r(x-r) & r < x \leq r+1, \\ f(t,x,y) & -r \leq x \leq r, \\ f(t,r,y) + M_r(x+r) & -r-1 \leq x < -r, \\ f(t,r,y) - M_r & x < -r-1. \end{cases}$$

As $f^*(t, r+1, 0) = f(t,r,0) + M_r > 0$, $r+1$ is a strict upper solution of the problem

$$(\Phi(x'))' = f^*(t,x,\Phi(x')) \tag{8}$$

$$x'(0) = 0, \qquad x(b) = \int_0^b x(s)dg(s) - k\Phi(x'(b)). \tag{2}$$

Similarly $-r-1$ is a strict lower solution of (8), (2).

As $|f^*(t,x,y)| < 2M_r$ f^* satisfies conditions of Lemma 2. Theorem 1 implies the existence of a solution of (8), (2) and

$$d(I - T^*, \Omega_{r+1,\rho}, 0) = 1 \qquad (\text{mod } 2),$$

where T^* is defined by the same formulas as T replacing f by f^*.

Let now

$$\Omega_l = \{x(t) \in \Omega_{r+1,\varrho}, \quad x < \beta\}, \qquad \Omega_u = \{x(t) \in \Omega_{r+1,\varrho}, \quad \alpha < x\}.$$

Then Theorem 2 and its proof imply $d(I - T^*, \Omega_l, 0) = d(I - T^*, \Omega_u, 0) = 1$ $(mod\ 2)$. Set $\Omega_m = \Omega_{r+1,\varrho} \setminus \left(\overline{\Omega_l \cup \Omega_u}\right)$. As $-r-1$, α are strict lower and $r+1$, β are strict upper solutions, Lemma 1 implies there is no solution $x \in \partial\Omega_m$.

The additivity of the degree yields

$$d(I - T^*, \Omega_m, 0) = 1 \qquad (mod\ 2).$$

Let $x(t) \in \Omega_m$ be a solution of (10), (11). We will prove that $|x(t)| < r$. Suppose $\exists t,\ x(t) > r$. Then the definition of Ω_m implies $\exists t_0,\ x(t_0) = r_0$. Moreover $(\Phi(x'))' = f^*(t,x,\Phi(x')) < 2M_r$. Integrating to t_1, the maximum of $x(t)$, we obtain $x'(t) < \Phi^{-1}(2M_r b)$ and $x(t) < x(t_0) + b\Phi^{-1}(2M_r b) = r$. Then $x(t) < r$ and similarly $x(t) > -r$. The definition of f^* implies $x(t)$ is a solution of (1), (2).

Multiplicity

The following two perturbation lemmas are based on Lemma 1.

Lemma 4. *Let* α *be a strict lower solution of the problem (1), (2).*

Set

$$f_\alpha(t,x,y) = \begin{cases} f(t,x,y) & x(t) > \alpha(t) \\ f(t,\alpha(t),y) & x(t) \leq \alpha(t). \end{cases}$$

Then each solution $x(t)$ *of*

$$(\Phi(x'))' = f_\alpha(t,x,\Phi(x')), \tag{9}$$

$$x'(0) = 0, \qquad x(b) = \int_0^b x(s)dg(s) - k\Phi(x'(b)), \tag{2}$$

is a solution of (1), (2).

Proof. Let $x(t)$ be a solution of (9), (2). Suppose that $m = \max(\alpha(t) - x(t)) \geq 0$. Then $\alpha(t) - m \leq x(t)$ and there is t_0 such that $\alpha(t_0) - m = x(t_0)$. As $\alpha(t) - m$ is a strict lower solution of (9), (2), we obtain a contradiction with Lemma 1.

Lemma 5. *Let β be a strict upper solution of the problem (1), (2).*

Set

$$f_\beta(t,x,y) = \begin{cases} f(t,x,y) & x(t) < \beta(t) \\ f(t,\beta(t),y) & x(t) \geq \beta(t). \end{cases}$$

Then each solution $x(t)$ of

$$(\Phi(x'))' = f_\beta(t,x,\Phi(x')), \quad (2)$$

is a solution of (1), (2).

The proof of the existence of multiple solutions is based on previous results.

Theorem 4. *Let*

(i) $\alpha < \beta$, $\alpha < \alpha_1$, $\alpha_1 \nleq \beta$, where α, α_1 are strict lower solutions and β is a strict upper solution of the problem (1), (2),

(ii) $\exists M > 0$ such that $|f(t,x,y)| \leq M$ for each $t \in I$, $\alpha(t) < x$, $y \in R$,

(iii) $G(b) < 1$.

Then the problem (1), (2) has at least two solutions.

Proof. Consider the problem (9), (2). Clearly $|f_\alpha| < M$. Theorem 2 implies the existence of a solution $x_1(t)$ of (9), (2), such that $\alpha < x_1 < \beta$, and Theorem 3 implies the existence of a solution $x_2(t)$ such that $\exists t_b \in I,\ x(t_b) > \beta(t_b)\}$. Lemma 4 implies x_1, x_2 are solutions of (1), (2).

Theorem 5. *Let*

(i) $\alpha < \beta$, $\beta_1 < \beta$, $\alpha \nleq \beta_1$, where α is a strict lower solution and β, β_1 are strict upper solutions of the problem (1), (2),

(ii) $\exists M > 0$ such that $|f(t,x,y)| \leq M$ for each $t \in I$, $x < \beta(t)$, $y \in R$,

(iii) $G(b) < 1$.

Then the problem (1), (2) has at least two solutions.

Example. Consider the boundary value problem for the equation

$$(\Phi(x'))' = f_1(t,x) + f_2(x') + h(t). \tag{10}$$

Assume that $f_1(t,x)$ is a continuous function such that

$$\lim_{x\to-\infty} f_1(t,x) = \infty, \qquad \lim_{x\to\infty} f_1(t,x) = -\infty,$$

uniformly for $t \in I$, and there are constants x_1, x_2, $x_1 < x_2$, such that $f_1(t,x_1) < f_1(t,x_2)$ for each $t \in I$. Further assume that f_2 is a continuous bounded function.

Then for each $h(t)$ there is $r > \max\{|x_1|,|x_2|\}$ sufficiently large, such that $-r, r$ are strict lower and upper solutions of (10), (2). Moreover, for each $h(t)$ such that $f_1(t,x_1) < h(t) < f_1(t,x_2)$, x_1 is a strict upper and x_2 a strict lower solution of (10), (2).

Then for each $h(t)$, $f_1(t,x_1) < h(t) < f_1(t,x_2)$, there are at least three solutions of the problem (10), (2).

For each $h(t)$, $f1(t,x_1) \leq h(t) \leq f_1(t,x_2)$, there are at least two solutions of the problem (10), (2).

Finally for each $h(t) \in C(I)$, exists a solution of (10), (2).

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Department of Mathematics, Faculty of Electrical Engineering STU,
812 19 Bratislava, Slovakia