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DECOMPOSITION THEOREMS FOR SESQUILINEAR FORMS

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1. Introduction

In our recent papers [4, 5] we presented general decomposition theorems for nonnegative sesquilinear forms that are common generalizations of several earlier results. For example, these generalize Ando's decomposition for positive operators [1] (see also [9]), Gudder's decomposition for positive functionals on Banach *-algebras [2], and Simon's decomposition for densely defined quadratic forms. Moreover, it turned out that there is a connection between the operator short and the Lebesgue decomposition of measures. The aim of this paper is to give a brief overview of this topic from an operator theoretic point of view.

2. Generalities

Let $\mathfrak{X}$ be a complex linear space and let $\mathfrak{t}$ be a nonnegative sesquilinear form (or shortly: *form*) on it. That is, $\mathfrak{t}$ is a mapping from the Cartesian product $\mathfrak{X}\times\mathfrak{X}$ to $\mathbb{C}$, which is linear in the first argument, antilinear in the second argument, and the corresponding quadratic form $\mathfrak{t}[\,\cdot\,]:\mathfrak{X}\to\mathbb{R}$

$$\forall x\in\mathfrak{X}:\quad \mathfrak{t}[x]:=\mathfrak{t}(x,x)$$

is nonnegative. The set $\mathcal{F}_+(\mathfrak{X})$ of forms is partially ordered with respect to the ordering

$$\mathfrak{t}\le\mathfrak{w}\iff\forall x\in\mathfrak{X}:\quad \mathfrak{t}[x]\le\mathfrak{w}[x].$$

If there exists a constant c such that $\mathfrak{t}\le c\cdot\mathfrak{w}$ then we say that $\mathfrak{t}$ is dominated by $\mathfrak{w}$ ($\mathfrak{t}\le_{\mathrm{d}}\mathfrak{w}$, in symbols). Since the square root of the quadratic form defines a seminorm on $\mathfrak{X}$, then the kernel of $\mathfrak{t}$

$$\ker\mathfrak{t}:=\big\{x\in\mathfrak{X}\;\big|\;\mathfrak{t}[x]=0\big\}$$

is a linear subspace of $\mathfrak{X}$. The Hilbert space $\mathscr{H}_{\mathfrak{t}}$ denotes the completion of the inner product space $\mathfrak{X}/_{\ker\mathfrak{t}}$ equipped with the natural inner product

$$\forall x,y\in\mathfrak{X}:\quad (x+\ker\mathfrak{t}\mid y+\ker\mathfrak{t})_{\mathfrak{t}}:=\mathfrak{t}(x,y).$$

The form $\mathfrak{t}$ is $\mathfrak{w}$-*absolutely continuous* if $\ker\mathfrak{w}\subseteq\ker\mathfrak{t}$, that is to say,

$$\forall x\in\mathfrak{X}:\quad \mathfrak{w}[x]=0\implies\mathfrak{t}[x]=0$$

in analogy with the well-known measure case. We say that the form $\mathfrak{t}$ is $\mathfrak{w}$-*closable* if

$$\big((\mathfrak{t}[x_n-x_m]\to0)\ \wedge\ (\mathfrak{w}[x_n]\to0)\big)\implies\mathfrak{t}[x_n]\to0$$

holds for all sequence $(x_n)_{n\in\mathbb{N}}\in\mathfrak{X}^{\mathbb{N}}$. The *singularity* of $\mathfrak{t}$ and $\mathfrak{w}$ means that

$$\forall\mathfrak{s}\in\mathcal{F}_+(\mathfrak{X}):\quad \big((\mathfrak{s}\le\mathfrak{t})\ \wedge\ (\mathfrak{s}\le\mathfrak{w})\big)\implies\mathfrak{s}=\mathfrak{o}.$$

In the following sections we present two fundamental results of decomposition theory of forms. The first one is the so-called short-type decomposition, which is a decomposition of a form into absolutely continuous and singular parts with respect to another one.

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3. Short-type decomposition

In our further considerations an essential role will be played by the concept of the short of a form, which is introduced as follows. Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be a linear subspace, and let $\mathfrak{t} \in \mathcal{F}_+(\mathfrak{X})$. Then the following formula defines a form

$$\forall x \in \mathfrak{X}: \quad \mathfrak{t}_{\mathfrak{Y}}[x] := \inf_{y \in \mathfrak{Y}} \mathfrak{t}[x-y] = \big\|(I-P)(x + \ker \mathfrak{t})\big\|_{\mathfrak{t}}^2.$$

Here P is the orthogonal projection from $\mathscr{H}_{\mathfrak{t}}$ onto the closure of $\mathfrak{Y}_{\mathfrak{t}} = \{y + \ker \mathfrak{t} \mid y \in \mathfrak{Y}\}$. The form $\mathfrak{t}_{\mathfrak{Y}}$ is the short of $\mathfrak{t}$ to the subspace $\mathfrak{Y}$.

Theorem 1. *Let $\mathfrak{t}, \mathfrak{w} \in \mathcal{F}_+(\mathfrak{X})$ be forms. Then there exists a short-type decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$. Namely,*

$$\mathfrak{t} = \mathfrak{t}_{\ker \mathfrak{w}} + (\mathfrak{t} - \mathfrak{t}_{\ker \mathfrak{w}}),$$

where the first summand is $\mathfrak{w}$-absolutely continuous and the second one is $\mathfrak{w}$-singular. Furthermore, $\mathfrak{t}_{\ker \mathfrak{w}}$ is maximal among those $\mathfrak{w}$-absolutely continuous forms that are majorized by $\mathfrak{t}$. The decomposition is unique precisely when $\mathfrak{t}_{\ker \mathfrak{w}}$ is dominated by $\mathfrak{w}$.

Corollary 2. *Let $\mathcal{E}$ be the complex linear space of measurable simple functions over the measurable space $(X, \mathcal{A})$. For a finite measure μ the following formula defines a form on $\mathcal{E}$:*

$$\forall \varphi, \psi \in \mathcal{E}: \quad \mathfrak{t}_\mu(\varphi, \psi) := \int_X \varphi \cdot \overline{\psi} \, \mathrm{d}\mu.$$

If μ and ν are finite measures, then μ is absolutely continuous with respect to ν precisely when $\mathfrak{t}_\mu$ is absolutely continuous with respect to $\mathfrak{t}_\nu$. Consequently, if $\mu = \mu_r + \mu_s$ is the unique Lebesgue decomposition of μ with respect to ν [10]*, then*

$$\mu_r(A) = \inf \Big\{ \int_A |1-\varphi|^2 \, \mathrm{d}\mu \;\Big|\; \varphi \in \mathcal{E}, \ \int_X |\varphi|^2 \, \mathrm{d}\nu = 0 \Big\}.$$

Remark 3. It was proved by Krein [3] that if $\mathscr{M}$ is a closed linear subspace of $\mathscr{H}$ and $A \in \mathbf{B}_+(\mathscr{H})$, then the set

$$\big\{ S \in \mathbf{B}_+(\mathscr{H}) \;\big|\; (S \le A) \ \wedge \ (\operatorname{ran} S \subseteq \mathscr{M}) \big\}$$

possesses a greatest element. This follows immediately from Theorem 1, and this is why we say that the form $\mathfrak{t}_{\mathfrak{Y}}$ is the *short of* $\mathfrak{t}$ *to the subspace* $\mathfrak{Y}$. Indeed, let $\mathfrak{t}(x,y) = (Ax \,|\, y)$ and consider the form $\mathfrak{t}_{\mathscr{M}^\perp}$. Since $\mathfrak{t}_{\mathscr{M}^\perp}$ is a bounded form, there exists a unique $S \in \mathbf{B}_+(\mathscr{H})$ such that $\mathfrak{t}_{\mathscr{M}^\perp}(x,y) = (Sx \,|\, y)$ and

$$\big(x \in \mathscr{M}^\perp \Rightarrow (Sx \,|\, x) = 0\big) \Rightarrow \mathscr{M}^\perp \subseteq \ker S \Rightarrow \operatorname{ran} S \subseteq \mathscr{M}.$$

The maximality of S follows from the maximality of $\mathfrak{t}_{\mathscr{M}^\perp}$.

4. Lebesgue-type decomposition

In this section we present the Lebesgue-type decomposition of forms.

Let J be the embedding operator from $\mathfrak{X}/_{\ker(\mathfrak{t}+\mathfrak{w})} \subseteq \mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ into $\mathfrak{H}_{\mathfrak{w}}$, defined by the identification

$$\forall x \in \mathfrak{X}: \quad x + \ker(\mathfrak{t}+\mathfrak{w}) \mapsto x + \ker \mathfrak{w}.$$

By setting

$$\mathfrak{S}(\mathfrak{t}, \mathfrak{w}) := \big\{ (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}} \;\big|\; \mathfrak{t}[x_n - x_m] \to 0, \mathfrak{w}[x_n] \to 0 \big\},$$

the kernel of J^{**} can be described by

$$\ker J^{**} = \Big\{ \lim_{n\to\infty} \big(x_n + \ker(\mathfrak{t}+\mathfrak{w})\big) \;\Big|\; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t}, \mathfrak{w}) \Big\}.$$

Let P stand for the orthogonal projection of $\mathfrak{H}_{\mathfrak{t}+\mathfrak{w}}$ onto $\{\ker J^{**}\}^{\perp}$, and define $\mathfrak{r} : \mathfrak{X} \to \mathbb{R}_+$ via the following formula:

$$\forall x \in \mathfrak{X} : \quad \mathfrak{r}[x] := \inf \big\{ \lim_{n\to\infty} \mathfrak{t}[x - x_n] \; \big| \; (x_n)_{n\in\mathbb{N}} \in \mathfrak{S}(\mathfrak{t}, \mathfrak{w}) \big\}.$$

For any $x \in \mathfrak{X}$ we have

$$\big\| P(x + \ker(\mathfrak{t} + \mathfrak{w})) \big\|^2_{\mathfrak{t}+\mathfrak{w}} = \mathfrak{r}[x] + \mathfrak{w}[x]$$

and

$$\big\| (I - P)(x + \ker(\mathfrak{t} + \mathfrak{w}) \big\|^2_{\mathfrak{t}+\mathfrak{w}} = \mathfrak{t}[x] - \mathfrak{r}[x].$$

In particular, both $\mathfrak{r}$ and $\mathfrak{t}-\mathfrak{r}$ are (quadratic) forms on $\mathfrak{X}$.

Theorem 4. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex linear space* $\mathfrak{X}$*. Then*

$$\mathfrak{t} = \mathfrak{r} + (\mathfrak{t} - \mathfrak{r})$$

is a Lebesgue-type decomposition of $\mathfrak{t}$ *with respect to* $\mathfrak{w}$*. That is,* $\mathfrak{r}$ *is closable with respect to* $\mathfrak{w}$*, and* $\mathfrak{t}-\mathfrak{r}$ *is singular with respect to* $\mathfrak{w}$*. Furthermore,*

$$\mathfrak{r} = \max\{\mathfrak{s} \in \mathcal{F}_+(\mathfrak{X}) \mid \mathfrak{s} \le \mathfrak{t},\ \mathfrak{s} \textit{ is } \mathfrak{w}\textit{-closable}\}.$$

That is, $\mathfrak{r}$ *is the maximum of all forms majorized by* $\mathfrak{t}$*, which are closable with respect to* $\mathfrak{w}$*.*

We will refer to $\mathfrak{r}$ (resp., to $\mathfrak{t}-\mathfrak{r}$) as the *regular part* (resp., the *singular part*) of $\mathfrak{t}$ with respect to $\mathfrak{w}$.

Corollary 5. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex) linear space* $\mathfrak{X}$*, and let* $\mathfrak{r}$ *denote the regular part of* $\mathfrak{t}$ *with respect to* $\mathfrak{w}$*. The following statements are equivalent:*

(i) $\mathfrak{t}$ *is* $\mathfrak{w}$*-closable;*
(ii) $\mathfrak{r} = \mathfrak{t}$*;*
(iii) $\ker J^{**} = \{0\}$*.*

Corollary 6. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be forms on the complex linear space* $\mathfrak{X}$*. Let* $\mathfrak{r}$ *stand for the regular part of* $\mathfrak{t}$ *with respect to* $\mathfrak{w}$*. Then for each* $x \in \mathfrak{X}$

$$\mathfrak{r}[x] = \inf \big\{ \liminf_{n\to\infty} \mathfrak{t}[x - x_n] \; \big| \; (x_n)_{n\in\mathbb{N}} \in \mathfrak{X}^{\mathbb{N}}, \mathfrak{w}[x_n] \to 0 \big\}.$$

Finally, we mention an application for positive operators. Let A and B be bounded positive operators on the Hilbert space $\mathscr{H}$. Applying our decomposition theorems to the forms

$$\mathfrak{t}_A(x, y) := (Ax \,|\, y) \quad \text{and} \quad \mathfrak{t}_B(x, y) = (Bx \,|\, y)$$

we gain the short-type decomposition

$$A = A_{\ll,B} + A_{\perp,B}$$

and the Lebesgue-type decomposition

$$A = \mathbf{D}_B A + (A - \mathbf{D}_B A)$$

of A with respect to B. If $\operatorname{ran} B$ is closed, then the shorted part $A_{\ll,B}$ coincides with the regular part $\mathbf{D}_B A$ in the sense of Ando [1], and therefore it is closable with respect to B. Furthermore, according to [9] we have the following characterization of closed range positive operators.

Theorem 7. *Let* B *be a bounded positive operator on the complex Hilbert space* $\mathscr{H}$*. Then the following are equivalent*

(i) $\operatorname{ran} B$ *is closed,*
(ii) $\forall A \in \mathbf{B}_+(\mathscr{H}) : \quad A_{\ll,B} \le_{\mathrm{d}} B$*,*
(iii) $\forall A \in \mathbf{B}_+(\mathscr{H}) : \quad \mathbf{D}_B A \le_{\mathrm{d}} B$*.*

If any of (i) – (iii) fulfills, then $\mathbf{D}_B A = A_{\ll,B}$ *for each positive operator* A*.*

5. Radon–Nikodym theorems

When we consider absolute continuity, there arises the natural question: can the regularity concept be characterized by a Radon–Nikodym type result? The following theorem answers this question in our general situation [8] (see also [6]).

Theorem 8. *Let $\mathfrak{t}$ and $\mathfrak{w}$ be forms on the complex linear space $\mathfrak{X}$. The following statements are equivalent:*

(i) $\mathfrak{t}$ *is* $\mathfrak{w}$*-closable,*

(ii) *There is a positive selfadjoint (in general, unbounded) operator T in $\mathfrak{H}_{\mathfrak{w}}$ such that $\mathfrak{X}/_{\ker \mathfrak{w}} \subseteq \operatorname{dom} T^{1/2}$ and*

$$\forall x \in \mathfrak{X}: \quad \mathfrak{t}[x] = \|T^{1/2}(x + \ker \mathfrak{w})\|^2_{\mathfrak{w}}.$$

Remark 9. Let $\mathcal{A}$ be a not necessarily unital *-algebra, and let w be a representable positive functional on it. That is to say, there exists a Hilbert space $\mathscr{H}_w$, a *-representation π_w of $\mathcal{A}$ to $\mathbf{B}(\mathscr{H}_w)$, and a cyclic vector ξ_w such that

$$\forall a \in \mathcal{A}: \quad w(a) = \big(\pi_w(a)\xi_w \,\big|\, \xi_w\big)_w.$$

Now, we have the following characterization: let v and w be representable functionals on $\mathcal{A}$. Then w is v-absolutely continuous in the sense of Gudder [2] precisely when there exists a positive selfadjoint operator W on $\mathscr{H}_v$ such that

$$\pi_v\langle\mathcal{A}\rangle\xi_v \subseteq \operatorname{dom} W$$

and

$$\forall a, b \in \mathcal{A}: \quad w(b^*a) = \big(W\pi_v(a)\xi_v \,\big|\, W\pi_v(b)\xi_v\big)_v.$$

The operators T and W in Theorem 8 and Remark 9 above might be called Radon–Nikodym derivatives.

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