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COMMUTING PAIRS OF ISOMETRIES – GENERALIZED POWERS

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1. Pairs defined by diagrams

The idea of pairs of isometries defined by a diagram appeared in [2]. Recall that a diagram is a set $J \subset \mathbb{Z}^2$ such that $(i,j) + J \subset J$ for any pair of nonnegative integers (i,j) where $(i,j) + J := \{(x+i, y+j) : (x,y) \in J\}$.

Definition 1. Let J be a diagram and $\mathcal{H}$ be a complex Hilbert space. Put

$$H = \bigoplus_{(i,j)\in J} H_{i,j} \text{ where } H_{i,j} = \mathcal{H}.$$

For $x = \{x_{i,j}\}_{(i,j)\in J} \in H$, define $y = \{y_{i,j}\}_{(i,j)\in J} \in H$ and $z = \{z_{i,j}\}_{(i,j)\in J} \in H$ by

$$y_{i,j} = \begin{cases} 0, & (i-1,j) \notin J \\ x_{i-1,j}, & (i-1,j) \in J, \end{cases} \quad \text{and} \quad z_{i,j} = \begin{cases} 0, & (i,j-1) \notin J, \\ x_{i,j-1}, & (i,j-1) \in J. \end{cases}$$

Define isometries V_1 and V_2 on H by

$$V_1 x = y \in H, \quad V_2 x = z \in H.$$

We call them the *isometries defined by diagram J and space $\mathcal{H}$.*

If $\mathcal{H}$ in Definition 1 is one dimensional ($\mathcal{H} = \mathbb{C}e_{i,j}$), then the pair of isometries is called simple:

Definition 2. Let J be a diagram and $H := \bigoplus_{(i,j)\in J} \mathbb{C}e_{i,j}$ where $\{e_{i,j}\}_{(i,j)\in J}$ are orthonormal. A pair of isometries $V_1, V_2 \in L(H)$ defined by $V_1 e_{i,j} = e_{i+1,j}$, $V_2 e_{i,j} = e_{i,j+1}$ for all $(i,j) \in J$ is called a *simple pair of isometries given by diagram J.*

2. Generalized powers

A special type of pairs of isometries are generalized powers defined in [1]. We precede the formal definition by giving some properties and background of the idea of generalized powers. For every such pair it holds equality $V_1^m = UV_2^n$ for some positive integers m, n and a unitary operator U. The name "generalized powers" follows from a generalization of the example: $V_1 = V^n, V_2 = V^m$ where V is a unilateral shift. In the example we have $V_1^m = V_2^n$ and $U = I$. An example of generalized powers which is the most different from the above one is when U is a bilateral shift. For respective wandering vectors such pair turns out to be a pair of generalized powers but also a pair defined by a diagram. We make a construction of such a pair starting from a diagram. Such a diagram has to be periodic in the following sense [1]:

Definition 3. The diagram J is periodic if there are positive numbers m, n such that for $J_0 := (\{0, 1, \ldots, m-1\} \times \mathbb{Z}) \cap J$ and $J_k = J_0 + k(m, -n) := \{(i + km, j - kn) : (i,j) \in J_0\}$ it holds $J = \sum_{k\in\mathbb{Z}} J_k$, where J_k are disjoint for different k. Then set J_0 is called a period of a diagram.

We now give a precise definition of generalized powers.

Definition 4. Let it be given:

(1) a periodic diagram $J = \sum_{k\in\mathbb{Z}} J_k$ with numbers m, n,

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(2) unitary operator $U \in L(\mathcal{H})$ and $e \in \mathcal{H}$ such that $\bigvee\{U^n e : n \in \mathbb{Z}\} = \mathcal{H}$.

Define:

(1) Hilbert space $H := \bigoplus_{(i,j)\in J_0} H_{i,j}$ where $H_{i,j} = \mathcal{H}$,
(2) $\hat{U} \in L(H)$ where $(\hat{U} \oplus_{(i,j)\in J_0} x_{i,j}) = \oplus_{(i,j)\in J_0} U(x_{i,j})$,
(3) $e_{i,j} \in H$ a vector such that $P_{H_{k,l}} e_{i,j} = e$ for $(k,l) = (i,j)$ and 0 otherwise, where $(i,j) \in J_0$,
(4) $e_{i+km,j-kn} = \hat{U}^k e_{i,j}$ for $(i,j) \in J_0$ and $k \in \mathbb{Z}$,
(5) $V_1(e_{i,j}) = e_{i+1,j}$, and $V_2(e_{i,j}) = e_{i,j+1}$.

Then operators V_1, V_2 are called *a pair of generalized powers given by a unitary operator* $\hat{U} \in L(H)$ *and a diagram* J.

Proposition 5. *Generalized powers are pairs of unilateral shifts.*

References

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