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ON DECOMPOSITIONS OF ISOMETRIES DEFINED BY A SZEGÖ PROPERTY

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Abstract. We consider a Szegö property which defines a unique Szegö decomposition of a measure. The Szegö decomposition of a measure generates Szegö decomposition of an isometry via elementary measures in a similar way like Lebesgue decomposition. However, Szegö decomposition of an isometry is not unique. Therefore, a class of Szegö type decompositions of an isometry is considered. Two such decompositions are defined. We give a few examples, describe some properties and applications.

1. Szegö property

A Borel measure μ on the σ-algebra of all Borel subsets of the unit circle $\mathcal{B}(\mathbb{T})$ is called a *Szegö measure*, if for any $\omega \in \mathcal{B}(\mathbb{T})$ the inclusion $\chi_\omega L^2(\mu) \subset H^2(\mu)$ implies $\mu(\omega) = 0$. We have the following (see [4]):

Proposition 1. *A measure μ is a Szegö measure if and only if*

(1) *μ is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{T}$,*
(2) *$\log \frac{d\mu}{dm}$ is Lebesgue integrable.*

The measure μ is *Szegö singular* if $H^2(\mu) = L^2(\mu)$. Moreover, each Borel regular measure μ on $\mathbb{T}$ has a unique decomposition

$$\mu = \chi_\omega \mu + \chi_{\mathbb{T}\setminus\omega}\mu,$$

where ω is a μ-essentially unique Borel set, $\chi_\omega\mu$ is a Szegö measure and $\chi_{\mathbb{T}\setminus\omega}\mu$ is Szegö singular. The proposition above shows relations between Szegö and Lebesgue decompositions. Every singular measure is Szegö singular and every Szegö measure is absolutely continuous to the Lebesgue measure. However, there are Szegö singular measures which are absolutely continuous to the Lebesgue measure.

2. Szegö decompositions of an isometry

Let $B(H)$ denote the algebra of all bounded linear operators on a complex Hilbert space H. For a given isometry $V \in B(H)$ denote by $H = H_u \oplus H_s$ its Wold decomposition and by E a spectral measure of its minimal unitary extension. For every $x \in H$ the mapping $\mu_x : \mathcal{B}(\mathbb{T}) \ni \omega \to \langle E(\omega)x, x\rangle$ is a positive Borel measure. The measure μ_x is called an *elementary measure* of x (and V). The Lebesgue decomposition of an isometry $V \in L(H)$ stands for $H = H_{sing} \oplus H_{ac}$ where H_{sing}, H_{ac} reduce V and measure μ_x is singular for $x \in H_{sing}$ and μ_x is absolutely continuous to the Lebesgue measure for $x \in H_{ac}$. Similar idea is used in a definition of a Szegö decomposition of an isometry. However, since a Szegö property is not preserved by a linear combination of vectors a linear span is used in the definition:

Definition 2. We call an isometry $V \in B(H)$ a *Szegö isometry* if H is spanned by vectors which elementary measures are Szegö. We call an isometry $V \in B(H)$ a *Szegö singular isometry* if an elementary measure of any vector is Szegö singular.

We say that a decomposition $V = V_1 \oplus V_2$ is a *Szegö-type decomposition* if V_1 is a Szegö singular isometry and V_2 is a Szegö isometry.

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A Szegö singular isometry is a unitary operator and a unilateral shift is a Szegö isometry. A class of Szegö singular isometries is bigger that a class of singular isometries. However, not all unitary isometries are Szegö singular. In conclusion $H_{sing} \subset H_{Sz-sing} \subset H_u$ and each inclusion can be proper where $H_{Sz-sing}$ denote a subspace reducing an isometry to a Szegö singular isometry.

There are different Szegö-type decompositions of an isometry. Natural examples introduced in [1] and [2] base on so-called wandering vectors. Recall that a vector w is called *wandering* if and only if $\langle V^n w, w\rangle = 0$, for all $n \in \mathbb{Z}_+$. It turns out that elementary measure of a wandering vector is a Szegö measure. Let us characterize Szegö singular isometries.

Proposition 3. *Let $V \in B(H)$ be an isometry. Let $L \subset H$ be a reducing subspace for V. Then each invariant (for V) subspace of L is reducing if and only if each elementary measure μ_x for $x \in L$ is Szegö singular.*

Now we introduce two Szegö type decompositions.

First, let us consider the reducing subspace $H_0 := H \ominus H_w$, where H_w is the subspace spanned by all wandering for V vectors. By [1] subspaces H_w and H_0 are reducing. Since an elementary measure of a wandering vector is Szegö then it can be showed that the decomposition $H = H_0 \oplus H_w$ is a Szegö-type decomposition. Such a decomposition will be called *I Szegö-type decomposition.*

For the second decomposition, note that since wandering vectors have elementary measures Szegö then a unilateral shift and a bilateral shift are Szegö isometries. Consider the decomposition $H = H_1 \oplus H_2$ such that H_2 reduce V to a direct sum of unilateral and bilateral shifts, and H_1 does not contain any wandering vector. Then $H = H_1 \oplus H_2$ generates a Szegö-type decomposition of V. This decomposition is not unequivocal. Thus we define the subspace

$$H_{ns} := \bigcap \{H_1 : H \ominus H_1 \text{ reduce } V \text{ to a direct sum of unilateral and bilateral shifts}\}.$$

For the orthogonal complement we have

$$(H_{ns})^{\perp} := \bigvee \{H_2 : H_2 \text{ reduce } V \text{ to a direct sum of unilateral and bilateral shifts}\}.$$

The decomposition $H = H_{ns} \oplus (H_{ns})^{\perp}$ is a Szegö-type decomposition of V. It will be called *II Szegö-type decomposition.*

For the example that justifies contradistinction of above decompositions let us first show some calculation. Set $S \in B(\bigoplus_{n=0}^{\infty} \ell^2)$ a unilateral shift of infinite multiplicity:

$$S\left(\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}\right) = \begin{bmatrix} 0 & a_{11} & a_{12} & \cdots \\ 0 & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Let $\{c_n\}_{n\geq 0}$ be a sequence of nonnegative numbers such that $\sum_{n\geq 0} c_n < \infty$.

Then

$$v = \begin{bmatrix} \sqrt{c_1} & -\sqrt{c_1} & 0 & \cdots \\ \sqrt{c_2} & 0 & -\sqrt{c_2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is well defined vector in $\bigoplus_{n=0}^{\infty} \ell^2$ such that $\langle S^n v, v\rangle = -c_n$ for any positive n. For a non unitary isometry $V \in L(H)$ if $x \in H_u$ is such that $\{\langle V^n x, x\rangle\}_{n\geq 0}$ is a convergent sequence of positive numbers then for v as above where $c_n = \langle V^n x, x\rangle$ the vector $x + v$ is wandering. Since $v \in H_s \subset H_w$ then $x \in H_w$.

Example 4. Denote $\mathbb{T}_+ := \{z \in \mathbb{T} : \Im z > 0\}$ and μ a Lebesgue measure on $\mathbb{T}_+$. Set $H = L^2(\mu) \oplus H^2(m)$ and denote by $V \in B(H)$ an operator of multiplication by $''z''$. Then it is easy to see that $H_{ns} = L^2(\mu)$. There is a vector $u \in L^2(\mu)$ such that $\{\langle V^n u, u\rangle\}_{n\geq 0}$ is a convergent sequence of positive numbers. Therefore $u + v$ is a wandering vector where v is a vector constructed as it was described. Consequently H_w is bigger that $H^2(\mu)$. In fact it can be showed that $H_w = H$ and $H_0 = \{0\}$.

I and *II* Szegö type decompositions are related by the inclusion $H_0 \subset H_{ns}$ which can be proper.

3. Application

Some motivation of investigating Szegö property was a description of a semigroup of isometries. By the canonical multiple Wold decomposition of a pair of isometries $V_1, V_2 \in L(H)$ we denote a decomposition $H = H_{uu}\oplus H_{us}\oplus H_{su}\oplus H_{ss}$, where $V_1|_{H_{uu}}, V_2|_{H_{uu}}, V_1|_{H_{us}}, V_2|_{H_{su}}$ are unitary operators and $V_1|_{H_{ss}}, V_2|_{H_{ss}}, V_2|_{H_{us}}, V_1|_{H_{su}}$ are unilateral shifts. Such a decomposition does not always exist. In the general case there is additional summand $H = H_{uu}\oplus H_{us}\oplus H_{su}\oplus H_{ss}\oplus H_{ncan}$, where H_{ncan} reduces V_1, V_2 to a non-canonical pair. A pair of isometries is called non-canonical if it can not be non-trivially reduced to a pair having a canonical decomposition. In other words for any $L \subset H_{ncan}$ reducing V_1, V_2 at least one of $V_1|_L, V_2|_L$ has non-trivial Wold decomposition as a single isometry. Assume isometries commute and set $H^i_{sing} \oplus H^i_{ac}$ the Lebesgue decomposition of an isometry V_i for $i = 1, 2$. Since the Lebesgue decomposition is hyperreducing it can be showed that $H_{ncan} \subset H^1_{ac} \cap H^2_{ac}$. It turns out that I Szegö decomposition is also hyperreducing. Therefore there is a decomposition

Theorem 5. *Let $V_1, V_2 \in B(H)$ be commuting isometries. Then there is*

$$H = H_{00} \oplus H_{w0} \oplus H_{0w} \oplus H_{ww}$$

into subspaces reducing V_1, V_2 where:

- *H_{00}, H_{0w} are of type H_0 for V_1 and H_{00}, H_{w0} are of type H_0 for V_2,*
- *H_{w0}, H_{ww} are linearly spanned by vectors wandering for V_1 and H_{0w}, H_{ww} are linearly spanned by vectors wandering for V_2.*

Subspaces H_{00}, H_{0w}, H_{w0} reduce both isometries to canonical pairs because in every such a pair at least one operator is unitary. Consequently $H_{ncan} \subset H_{ww}$. The I Szegö decomposition remains in a very close relation with the Lebesgue decomposition. Although this decompositions are not equal, in the interesting cases they give similar results. Therefore $H_{ncan} \subset H_{ww}$ refines $H_{ncan} \subset H^1_{ac} \cap H^2_{ac}$ by giving another description rather that significantly smaller superspace of H_{ncan}.

Unitary operator $U \in B(H)$ is called a *span of bilateral shifts* if $H = \overline{\bigvee H_n}$ where $V|_{H_n}$ is a bilateral shift for every n. As a consequence of $H_{ncan} \subset H_{ww}$ we have the following:

Corollary 6. *If $V_1, V_2 \in B(H)$ is a completely non-canonical pair then unitary extension of each isometry is a span of bilateral shifts.*

A span of bilateral shifts may not be a bilateral shift. As an example take operator of multiplication by independent variable $''z''$ on the space $L^2(\mathbb{T}_+) \oplus L^2(\mathbb{T}_-) \oplus L^2(\mathbb{T}_+)$ where $\mathbb{T}_+, \mathbb{T}_-$ are upper and bottom half-circles respectively. However, if a span of bilateral shifts contains a bilateral shift of multiplicity ∞ then it is a bilateral shift. Recall from [3] that

Theorem 7. *Let $V_1, V_2 \in B(H)$ be commuting isometries such that $\dim\ker V_1^* < \infty$ then Wold decomposition of V_2 reduces also V_1.*

Assume that $V_1, V_2 \in B(H)$ are commuting isometries such that $H = H_{ncan}$. By above theorem at least one of wandering subspaces is infinite dimensional. If $\dim\ker V_1^* < \infty$ then by the above theorem H^2_u (subspace reducing V_2 to a unitary operator) reduces both isometries. On the other hand a unitary operator $(V_2|_{H^2_u})$ commuting with other isometry has a canonical multiple Wold decomposition. Therefore by assumption $H = H_{ncan}$ we conclude that $H^2_u = \{0\}$ and consequently V_2 is a unilateral shift of infinite multiplicity. In conclusion if $H = H_{ncan}$ then unitary extension of at least one of isometries is a bilateral shift of infinite multiplicity.

References

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