10th Workshop Functional Analysis and its Applications, September 7–12, 2015, Kočovce, Slovakia, 39–41

SPECTRAL REPRESENTATION FOR A CLASS OF LAURENT OPERATORS

WITOLD MAJDAK AND EWELINA ZALOT

Abstract. We describe some spectral representations for a class of non-self-adjoint banded Jacobi-type matrices. Our results extend those obtained by P.B. Naïman for (two-sided infinite) periodic tridiagonal Jacobi matrices.

1. Introduction

First we recall some key definitions and facts from [2]. Let us consider a finite family of smooth non-intersecting curves on the complex plane. We enumerate them as $\Gamma_1, \ldots, \Gamma_d$, impose an orientation on each of them, and denote by α and β the beginning of Γ_1 and the end of Γ_d, respectively. We set $\Gamma^\circ = \Gamma_1 \cup \ldots \cup \Gamma_d$ and introduce the order relation $\prec$ on Γ° as follows: for $\lambda, \mu \in \Gamma^\circ$, we write $\lambda \prec \mu$ if λ and μ are on the same curve, whereas λ lies earlier than μ in accordance with the fixed orientation, or $\lambda \in \Gamma_i$ and $\mu \in \Gamma_j$ for some $i < j$ $(i, j = 1, \ldots, d)$. We denote by Γ the closure of Γ° and next in a natural way extend the order $\prec$ to Γ distinguishing, to avoid the ambiguity, the points which are simultaneously beginnings and ends of the corresponding curves.

Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$.

Definition 1. An operator-valued function $E : \Gamma \to \mathcal{B}(\mathcal{H})$ is a *resolution of the identity* if:

(i) $E(\lambda)\mathcal{H} \subset E(\mu)\mathcal{H}$ for all $\lambda, \mu \in \Gamma$ such that $\lambda \prec \mu$
(ii) for each $\lambda \in \Gamma$, $E(\lambda)$ is an orthogonal projection,
(iii) $E(\alpha) = 0$ and $E(\beta) = I$.

Take an operator $A \in \mathcal{B}(\mathcal{H})$ (which, in general, is non-selfadjoint). Assume henceforth that $\sigma(A) = \Gamma$, where $\sigma(A)$ stands for the spectrum of A. We provide the following definitions ([2]).

Definition 2. A resolution of the identity E is called a *spectral resolution* of the operator $A \in \mathcal{B}(\mathcal{H})$ if:

(i) for each $\lambda \in \Gamma$, the space $E(\lambda)\mathcal{H}$ is invariant for A,
(ii) for each $\lambda \in \Gamma$,

$$\sigma(AE(\lambda)) = \overline{\{\mu \in \Gamma : \mu \prec \lambda\}} \quad \text{and} \quad \sigma(A(I - E(\lambda))) = \overline{\{\mu \in \Gamma : \lambda \prec \mu\}}.$$

Definition 3. An operator-valued function $F : \Gamma \to \mathcal{B}(\mathcal{H})$ is said to be a *skew resolution of the identity* if:

(i) for each $\lambda \in \Gamma$, $F(\lambda)$ is a projection (i.e. $F^2(\lambda) = F(\lambda)$),
(ii) $F(\alpha) = \lim_{\lambda \downarrow \alpha} F(\lambda) = 0$ and $F(\beta) = \lim_{\lambda \uparrow \beta} F(\lambda) = I$,
(iii) $F(\lambda)\mathcal{H} \subset F(\mu)\mathcal{H}$ and $(I - F(\mu))\mathcal{H} \subset (I - F(\lambda))\mathcal{H}$ for all $\lambda, \mu \in \Gamma$ such that $\lambda \prec \mu$,
(iv) there exist positive real numbers m and M such that for each $g \in \mathcal{H}$ and each division $\{\Delta_1, \ldots, \Delta_s\}$ of Γ on open intervals $\Delta_k = (\alpha_k, \beta_k)$ $(k = 1, \ldots, s)$, we have

$$m \sum_{k=1}^{s} \|F(\Delta_k)g\|^2 \le \|g\|^2 \le M \sum_{k=1}^{s} \|F(\Delta_k)g\|^2,$$

2010 *Mathematics Subject Classification.* 47A67, 47A75.

where $F(\Delta_k) := F(\beta_k) - F(\alpha_k)$.

Definition 4. A skew resolution of the identity F is called a *skew spectral resolution* of $A \in \mathcal{B}(\mathcal{H})$ if:

(i) for each $\lambda \in \Gamma$, the spaces $F(\lambda)\mathcal{H}$ and $(I - F(\lambda))\mathcal{H}$ are invariant for A,
(ii) for each $\lambda \in \Gamma$,

$$\sigma(AF(\lambda)) = \overline{\{\mu \in \Gamma : \mu \prec \lambda\}} \quad \text{and} \quad \sigma(A(I - F(\lambda))) = \overline{\{\mu \in \Gamma : \lambda \prec \mu\}},$$

(iii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each interval $\Delta \subset \Gamma$ of length less than δ, and all $f \in F(\Delta)\mathcal{H}$ and $\lambda \in \Delta$, we have

$$\|Af - \lambda f\| < \varepsilon.$$

If F is a skew-spectral resolution of an operator A on $\mathcal{H}$, then

$$Af = \int_\Gamma \lambda \, \mathrm{d}F(\lambda) f, \quad f \in \mathcal{H},$$

where the integral above is understood in the Riemann sense, i.e. it is the limit of the sums of the form $\sum_{k=1}^{s} \lambda_k F(\Delta_k) f$.

2. Spectral resolutions of Laurent operators

Let $A : l^2(\mathbb{Z}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ be a Laurent operator defined by

$$(Au)_n = \sum_{j=-\infty}^{\infty} A_j u_{n-j}, \quad n \in \mathbb{Z},$$

for $u = (u_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{C}^d)$, where $A_j \in \mathbb{C}^{d\times d}$ $(j \in \mathbb{Z})$ and $\sum\limits_{j=-\infty}^{\infty} |A_j| < \infty$. Let $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. With regard to the terminology used in [1], the matrix-valued function

$$\mathcal{A}(\zeta) = \sum_{k=-\infty}^{\infty} A_k \zeta^k, \quad \zeta \in \mathbb{T},$$

is called the *symbol* of the operator A.

Let us assume that $\sigma(A) = \Gamma$, where Γ is the closure of a finite sum of curves as in Section 1. We will show that a spectral representation of A can be derived from that of the symbol $\mathcal{A}(\zeta)$. For this purpose we consider two cases: first a more general one, when the symbol $\mathcal{A}(\zeta)$ is triangularizable, and then a particular one, when it is diagonalizable.

Case 1. Spectral resolution for a Laurent operator

For a fixed $\zeta \in \mathbb{T}$, $\mathcal{A}(\zeta)$ is a $d \times d$ complex matrix. In view of the Schur theorem there exist:

- a triangular matrix $T(\zeta)$,
- a unitary matrix $U(\zeta)$

such that

$$\mathcal{A}(\zeta) = U(\zeta) T(\zeta) (U(\zeta))^*.$$

Next, we consider the operator $T : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ defined by

$$(T\psi)(\zeta) = T(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d), \zeta \in \mathbb{T}.$$

We call T a *canonical operator* of A.

Theorem 5. *Let $A : l^2(\mathbb{Z}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ be a Laurent operator, $\mathcal{A}(\zeta)$ its symbol, and $T(\zeta)$ a Schur triangularization of $\mathcal{A}(\zeta)$ $(\zeta \in \mathbb{T})$. Then:*

(i) *the operator A is unitarily equivalent to the canonical operator $T : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ (more precisely, $A = \mathcal{U} T \mathcal{U}^{-1}$, where $\mathcal{U} : L^2(\mathbb{T}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ is a unitary operator),*

(ii) *the operator-valued function $\mathcal{E}$ defined by*

$$\mathcal{E}(\lambda) = \mathcal{U}E(\lambda)\mathcal{U}^{-1}, \quad \lambda \in \sigma(A),$$

is a spectral resolution of A.

Case 2. Skew spectral resolution for a Laurent operator

Assume that for each $\zeta \in \mathbb{T}$ the matrix $\mathcal{A}(\zeta)$ has all simple eigenvalues $\lambda_k(\zeta)$ $(k = 1, \dots, d)$. Then each matrix $\mathcal{A}(\zeta)$ is diagonalizable. More precisely, for $\zeta \in \mathbb{T}$, we can find:

- a diagonal matrix $D(\zeta)$,
- an invertible matrix $V(\zeta)$

such that

$$\mathcal{A}(\zeta) = V(\zeta)D(\zeta)(V(\zeta))^{-1}.$$

Let $D : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ be the operator given by

$$(D\psi)(\zeta) = D(\zeta)\psi(\zeta), \quad \psi \in L^2(\mathbb{T}, \mathbb{C}^d), \zeta \in \mathbb{T}.$$

Theorem 6. *Let $A : l^2(\mathbb{Z}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ be a Laurent operator and $\mathcal{A}(\zeta)$ $(\zeta \in \mathbb{T})$ its symbol. Suppose that for each $\zeta \in \mathbb{T}$ the matrix $\mathcal{A}(\zeta)$ has simple eigenvalues only. Then:*

(i) *A is similar to the operator $D : L^2(\mathbb{T}, \mathbb{C}^d) \to L^2(\mathbb{T}, \mathbb{C}^d)$ (more precisely, $A = WDW^{-1}$, where $W : L^2(\mathbb{T}, \mathbb{C}^d) \to l^2(\mathbb{Z}, \mathbb{C}^d)$ is an invertible bounded operator),*

(ii) *the operator-valued function $\mathcal{F}$ defined by*

$$\mathcal{F}(\lambda) = WF(\lambda)W^{-1}, \quad \lambda \in \sigma(A),$$

is a skew spectral resolution of A,

(iii) *A has an integral representation*

$$Af = \int_{\Gamma} \lambda \, \mathrm{d}\mathcal{F}(\lambda) f, \quad f \in l^2(\mathbb{Z}, \mathbb{C}^d).$$

All results presented above can be found in [3].

References

[1] A. Böttcher, B. Silbermann, *Analysis of Toeplitz Operators*, Springer–Verlag, Heidelberg, 1990.

[2] P.B. Naïman, *On the spectral theory of the non-symetric periodic Jacobi matrices*, Notes of the Faculty of Math. and Mech. of Kharkov's State University and of Kharkov's Math. Society **30** (1964), 138–151 [in Russian].

[3] E. Zalot, W. Majdak, *Spectral representations for a class of banded Jacobi-type matrices*, Opuscula Math. **34** (2014), 871-887.

AGH University of Science and Technology, Faculty of Applied Mathematics, al. A. Mickiewicza 30, 30-059 Krakow, Poland

E-mail address: majdak@agh.edu.pl, zalot@agh.edu.pl