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HAUSDORFF DIMENSION OF UNIVOQUE SETS AND DEVIL'S STAIRCASE

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We report on some joint works with P. Erdős, I. Joó, P. Loreti, M. de Vries, Derong Kong and Wenxia Li. For an overview and references we refer to [3].

Fix a positive integer M and an *alphabet* $\{0, 1, \dots, M\}$. By a *sequence* we mean an element $c = (c_i)$ of $\{0, 1, \dots, M\}^\infty$.

Given a real *base* $q > 1$, by an *expansion* of a real number x we mean a sequence $c = (c_i)$ satisfying the equality

$$\pi_q(c) := \sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

Expansions of this type in *non-integer* bases have been extensively investigated since a pioneering paper of Rényi. One of the striking features of such bases is that generically a number has a continuum of different expansions, a situation quite opposite to that of integer bases; see, e.g., [4] and Sidorov. However, surprising unique expansions have also been discovered by Erdős, Horváth and Joó, and they have stimulated many works during the last 25 years.

Let us denote by $\mathcal{U}_q$ the set of numbers x having a unique expansion and by $\mathcal{U}_q'$ the set of the corresponding expansions. The topological and combinatorial structure of these sets have been described in [1]. The present paper is a natural continuation of this work, concerning the measure-theoretical aspects.

Daróczy and Kátai have determined the Hausdorff dimension of $\mathcal{U}_q$ when $M = 1$ and q is a Parry number. Their results were extended by Kallós and Kátai, Glendinning and Sidorov, Kong et al., and in [2].

We recall from [5] and [6] that there exists a smallest base $1 < q' < M + 1$ (depending on M) in which $x = 1$ has a unique expansion: the so-called *Komornik–Loreti constant.*

We also recall two theorems on the *dimension function*

$$D(q) := \dim_H \mathcal{U}_q, \quad 1 < q < \infty,$$

obtained respectively by Glendinning and Sidorov, and by Derong Kong, Michel Dekking and Wenxia Li:

Theorem 1. *The function D vanishes in $(1, q']$, and $D > 0$ in (q', ∞). Its maximum $D(q) = 1$ is attained only in $q = M + 1$.*

It follows from this theorem that $\mathcal{U}_q$ is a (Lebesgue) null set for all $q \neq M + 1$, while $\mathcal{U}_{M+1} \subseteq [0, 1]$ has measure one because its complementer set is countable in $[0, 1]$. Since $\overline{\mathcal{U}_q} \setminus \mathcal{U}_q$ is countable for each q (see [1]), the same properties hold for $\overline{\mathcal{U}_q}$ as well.

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Theorem 2. *For almost all $q > 1$, $\mathcal{U}'_q$ is a subshift, and*

$$D(q) = \frac{h(\mathcal{U}'_q)}{\log q}, \tag{1}$$

where $h(\mathcal{U}'_q)$ denotes the topological entropy of $\mathcal{U}'_q$.

Furthermore, the function D is differentiable almost everywhere.

We recall that

$$h(\mathcal{U}'_q) = \lim_{n\to\infty} \frac{\log |B_n(\mathcal{U}'_q)|}{n} = \inf_{n\ge 1} \frac{\log |B_n(\mathcal{U}'_q)|}{n} \tag{2}$$

when $\mathcal{U}'_q$ is a subshift, where $B_n(\mathcal{U}'_q)$ denotes the set of different initial words of length n occurring in the sequences $(c_i) \in \mathcal{U}'_q$, and $|B_n(\mathcal{U}'_q)|$ means the cardinality of $B_n(\mathcal{U}'_q)$. (Unless otherwise stated, in this paper we use base two logarithms.)

We will complete and improve Theorems 1 and 2 in Theorems 3, 4 and 7 below.

Theorem 3. *The formula* (1) *is valid for* all $q > 1$.

We recall from [1] that $\mathcal{U}'_q$ is not always a subshift. Theorem 3 states in particular that the limit in (2) exists even if $\mathcal{U}'_q$ is not a subshift, and it is equal to the infimum in (2).

Theorem 4. *The function D is continuous, and has a bounded variation.*

Theorem 4 implies again that D is differentiable almost everywhere. In order to describe its derivative first we establish some results on general β-expansions and on univoque bases.

Following Rényi we denote by $\beta(q) = (\beta_i(q))$ the lexicographically largest expansion of $x = 1$ in base q. It is also called the greedy or β-expansion of $x = 1$ in base q.

Theorem 5. *Fix $1 < r \le M+1$ arbitrarily. For almost all $q \in (1, r)$ there exist arbitrarily large integers m such that $\beta_1(q)\cdots\beta_m(q)$ ends with more than $\log_r m$ consecutive zero digits.*

This theorem improves and generalizes [4, Theorem 2] concerning the case $M = 1$. In particular, our result implies that $\beta(q)$ contains arbitrarily large blocks of consecutive zeros for almost all $q \in (1, M+1]$. This was first established by Erdős and Joó for $M = 1$, and their result was extended by Schmeling for all M.

Next we denote by $\mathcal{U}$ the set of bases $q > 1$ in which $x = 1$ has a unique expansion, and by $\overline{\mathcal{U}}$ its closure. The elements of $\mathcal{U}$ are usually called *univoque bases*.

Theorem 6.

(i) *$\mathcal{U}$ and $\overline{\mathcal{U}}$ are (Lebesgue) null sets.*
(ii) *$\mathcal{U}$ and $\overline{\mathcal{U}}$ have Hausdorff dimension one.*

Parts (i) and (ii) were proved for $\mathcal{U}$ in case $M = 1$ by Erdős and Joó, and by Daróczy and Kátai, respectively. The case of $\overline{\mathcal{U}}$ hence follows because the set $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable (see [7]). Our proof of (ii) is shorter than the original one even for $M = 1$.

Finally, combining Theorems 1, 3, 4, 6 (i) and some topological results of [1] we prove that the dimension function is a natural variant of *Devil's staircase*:

Theorem 7.

(i) *D is continuous in $[q', \infty)$.*
(ii) *$D' < 0$ almost everywhere in (q', ∞).*
(iii) *$D(q') < D(q)$ for all $q > q'$.*

Remark. Compared to the classical Cantor–Lebesgue function, we have even $D' < 0$ instead of $D' = 0$ almost everywhere.

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