10th Workshop Functional Analysis and its Applications, September 7–12, 2015, Kočovce, Slovakia, 34–38

SYMMETRIES FOR AN INTEGRO-DIFFERENTIAL EQUATION IN A DISK

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ABSTRACT. We investigate the wave equation with an integral term in a disk. Our goal is to write the solution as Fourier series under a radial symmetric assumption on the data. The expression of the solution obtained allows us to get explicit Ingham type estimates, and hence reachability results.

In this note we will consider

$$u_{tt} - \triangle u + \beta \int_0^t e^{-\eta(t-s)} \triangle u(s,x,y)ds = 0\,, \quad t \geq 0\,,\ (x,y) \in \Omega, \tag{1}$$

where $\triangle$ denotes the Laplace operator in a circular disk Ω of radius R in $\mathbb{R}^2$ and $0 < \beta < \eta$. It's natural to use polar coordinates. Indeed, taking into account that in polar coordinates the Laplacian is given by

$$\triangle = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,,$$

we can rewrite equation (1) as:

$$u_{tt} - \frac{1}{r}\big(ru_r\big)_r - \frac{1}{r^2}u_{\theta\theta} + \frac{\beta}{r^2}\int_0^t e^{-\eta(t-s)}\Big(r\big(ru_r\big)_r + u_{\theta\theta}\Big)(s,r,\theta)ds = 0\,, \qquad t \geq 0\,,\ (r,\theta) \in \mathcal{D}, \tag{2}$$

where $\mathcal{D} = \{(r,\theta) : 0 < r < R\,,\ \theta \in [0,2\pi]\}$, and solve for u as a function of t, r and θ.

For the sake of completeness, we briefly recall standard argumentations. To determine the eigenvalues of the Laplacian, we have to solve

$$-\triangle u(r,\theta) = \lambda^2 u(r,\theta) \tag{3}$$

$$u(R,\theta) = 0 \tag{4}$$

To this end, we attempt separation of variables by writing

$$u(r,\theta) = \mathcal{R}(r)\Theta(\theta).$$

Then (3) becomes

$$r^2\frac{d^2\mathcal{R}}{dr^2}\Theta + r\frac{d\mathcal{R}}{dr}\Theta + \mathcal{R}\frac{d^2\Theta}{d\theta^2} + \lambda^2 r^2\mathcal{R}\Theta = 0\,.$$

If we divide by $\mathcal{R}\Theta$, then we obtain

$$\frac{r^2}{\mathcal{R}}\frac{d^2\mathcal{R}}{dr^2} + \frac{r}{\mathcal{R}}\frac{d\mathcal{R}}{dr} + \frac{1}{\theta}\frac{d^2\Theta}{d\theta^2} + \lambda^2 r^2 = 0\,. \tag{5}$$

The function Θ must be sinusoidal, that is

$$\frac{1}{\theta}\frac{d^2\Theta}{d\theta^2} = -n^2, \tag{6}$$

and hence, for $a_n \in \mathbb{C}$ we have

$$\Theta(\theta) = a_n e^{in\theta} + \overline{a_n}e^{-in\theta}. \tag{7}$$

2010 *Mathematics Subject Classification.* 45K05.

Key words and phrases. Integrodifferential wave equations, convolution kernels; Fourier series.

Plugging (6) into (5), we obtain

$$r^2\frac{d^2\mathcal{R}}{dr^2}+r\frac{d\mathcal{R}}{dr}+(\lambda^2r^2-n^2)\mathcal{R}=0\,, \tag{8}$$

with the boundary condition $\mathcal{R}(R)=0$. We can eliminate λ^2 from the previous equation by making a change of variables. Indeed, if we set $x=\lambda r$, then the equation (8) becomes

$$x^2\frac{d^2\mathcal{R}}{dx^2}+x\frac{d\mathcal{R}}{dx}+(x^2-n^2)\mathcal{R}=0\,, \tag{9}$$

which is called *Bessel's equation of order* n. A solution of (9) is given by

$$J_n(x)=\sum_{h=0}^{\infty}\frac{(-1)^h}{h!(h+n)!}\Big(\frac{x}{2}\Big)^{n+2h}\,, \tag{10}$$

which is called the *Bessel function of the first kind of order* n. It follows that a solution of (8) is given by $J_n(x)=J_n(\lambda r)$. The boundary condition $\mathcal{R}(R)=0$ is satisfied if

$$J_n(\lambda R)=0,$$

that is

$$\lambda=\frac{\lambda_{nk}}{R}\,,$$

where λ_{nk}, $k\in\mathbb{N}$, are the positive zeros of J_n.

We recall the following result, see [5, Section 6].

Theorem 1. *Let $\mathcal{A}: D(\mathcal{A})\subset H\to H$ be a self-adjoint positive linear operator on a Hilbert space H with dense domain $D(\mathcal{A})$. Assume that $\{\lambda_j\}_{j\ge1}$ is a strictly increasing sequence of eigenvalues for $\mathcal{A}$, with $\lambda_j>0$ and $\lambda_j\to\infty$, and the sequence $\{w_j\}_{j\ge1}$ of the corresponding eigenvectors constitutes an orthogonal basis for H.*

The general solution of equation

$$u''(t)+\mathcal{A}u(t)-\beta\int_0^t e^{-\eta(t-s)}\mathcal{A}u(s)ds=0\,,\quad t\ge0\,, \tag{11}$$

can be written as the following series

$$u(t)=\sum_{j=1}^{\infty}\big(R_je^{r_jt}+C_je^{i\omega_jt}+\overline{C_j}e^{-i\overline{\omega_j}t}\big)w_j\,,\qquad R_j\in\mathbb{R},\ C_j\in\mathbb{C}, \tag{12}$$

where $r_j\in\mathbb{R}$ and $\omega_j\in\mathbb{C}$ are defined by

$$\begin{aligned} r_j&=\beta-\eta+O\Big(\frac{1}{\lambda_j}\Big)\,,\\ \omega_j&=\sqrt{\lambda_j}+\frac{\beta}{2}\Big(\frac{3}{4}\beta-\eta\Big)\frac{1}{\sqrt{\lambda_j}}+i\frac{\beta}{2}+O\Big(\frac{1}{\lambda_j}\Big)\,. \end{aligned} \tag{13}$$

Let $H=L^2(\mathcal{D})$ be endowed with the scalar product and norm

$$\langle u,v\rangle:=\int_0^R\int_0^{2\pi}ru(r,\theta)v(r,\theta)\,drd\theta\,,\quad \|u\|:=\Big(\int_0^R\int_0^{2\pi}r|u(r,\theta)|^2\,drd\theta\Big)^{1/2}\quad u,v\in L^2(\mathcal{D})\,.$$

The operator $\mathcal{A}: D(\mathcal{A})\subset H\to H$ is defined by

$$D(\mathcal{A})=H^2(\mathcal{D})\cap H^1_0(\mathcal{D})$$

$$\mathcal{A}u=-\triangle u\qquad u\in D(\mathcal{A})\,.$$

It is well known that $\mathcal{A}$ is a self-adjoint positive operator on H with dense domain $D(\mathcal{A})$, the eigenvalues for $\mathcal{A}$ are $\big(\frac{\lambda_{nk}}{R}\big)^2$ and the corresponding eigenfunctions are $J_n\big(\frac{\lambda_{nk}}{R}r\big)e^{\pm in\theta}$, which form an orthogonal basis for $L^2(\mathcal{D})$.

In order to simplify notations, we will define J_{-n} to be the same as J_n whenever n is an integer:

$$J_{-n} := J_n, \quad \lambda_{-nk} := \lambda_{nk}\,, \qquad n \in \mathbb{N}\cup\{0\}, \quad k \in \mathbb{N}\,.$$

We are going to establish the result in the 2-D case

$$u_{tt} - \frac{1}{r}\big(ru_r\big)_r - \frac{1}{r^2}u_{\theta\theta} + \frac{\beta}{r^2}\int_0^t e^{-\eta(t-s)}\Big(r\big(ru_r\big)_r + u_{\theta\theta}\Big)(s,r,\theta)ds = 0\,, \qquad t\ge 0\,,\ (r,\theta)\in\mathcal{D}, \tag{14}$$

Therefore, thanks to (12) we have

$$u(t,r,\theta) = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}\Big(R_{nk}e^{r_{nk}t+in\theta} + C_{nk}e^{i(\omega_{nk}t+n\theta)} + \overline{C_{nk}}e^{-i(\overline{\omega_{nk}}t+n\theta)}\Big)J_n\Big(\frac{\lambda_{nk}}{R}r\Big)\,, \tag{15}$$

where $r_{nk}\in\mathbb{R}$ and $\omega_{nk}\in\mathbb{C}$ are defined by

$$r_{nk} = \beta - \eta + O\Big(\frac{1}{\lambda_{nk}^2}\Big)\,, \quad n\in\mathbb{N}$$

$$\Re\omega_{nk} = \frac{\lambda_{nk}}{R} + \frac{\beta}{2}\Big(\frac{3}{4}\beta - \eta\Big)\frac{R}{\lambda_{nk}} + O\Big(\frac{1}{\lambda_{nk}^2}\Big)\,, \quad \Im\omega_{nk} = \frac{\beta}{2} + O\Big(\frac{1}{\lambda_{nk}^2}\Big)\,, \quad n\in\mathbb{N}, \tag{16}$$

$$r_{-nk} := r_{nk} \qquad \omega_{-nk} := -\overline{\omega_{nk}}\,, \quad n\in\mathbb{N}.$$

The coefficients $R_{nk}\,, C_{nk}$ are complex numbers to determine and $\overline{R_{nk}} = R_{nk}$. If we impose the initial conditions

$$u(0,r,\theta) = f(r)e^{i\theta} + \overline{f(r)}e^{-i\theta}\,, \quad u_t(0,r,\theta) = 0\,, \tag{17}$$

then we obtain ($R=1$, $0<r<1$)

$$u(0,r,\theta) = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}\Big(R_{nk}e^{in\theta} + C_{nk}e^{in\theta} + \overline{C_{nk}}e^{-in\theta}\Big)J_n(\lambda_{nk}r) = f(r)e^{i\theta} + \overline{f(r)}e^{-i\theta} \tag{18}$$

$$\sum_{\substack{n=0\\ n\neq 1}}^{\infty}\sum_{k=1}^{\infty}\Big(R_{nk} + C_{nk} + \overline{C_{-nk}}\Big)J_n(\lambda_{nk}r)e^{in\theta} = 0 \tag{19}$$

$$\sum_{\substack{n=0\\ n\neq 1}}^{\infty}\sum_{k=1}^{\infty}\Big(R_{-nk} + C_{-nk} + \overline{C_{nk}}\Big)J_n(\lambda_{nk}r)e^{-in\theta} = 0 \tag{20}$$

$$R_{nk} + C_{nk} + \overline{C_{-nk}} = 0\,, \quad \forall n\in\mathbb{N}\,, n\neq 1\,, k\in\mathbb{N} \tag{21}$$

$$\sum_{k=1}^{\infty}\big(R_{1k} + C_{1k} + \overline{C_{-1k}}\big)J_1(\lambda_{1k}r)e^{i\theta} + \sum_{k=1}^{\infty}\big(R_{-1k} + C_{-1k} + \overline{C_{1k}}\big)J_1(\lambda_{1k}r)e^{-i\theta} = f(r)e^{i\theta} + \overline{f(r)}e^{-i\theta} \tag{22}$$

$$\sum_{k=1}^{\infty}\big(R_{1k} + C_{1k} + \overline{C_{-1k}}\big)J_1(\lambda_{1k}r) = f(r) \tag{23}$$

$$\sum_{k=1}^{\infty}\big(R_{-1k} + C_{-1k} + \overline{C_{1k}}\big)J_1(\lambda_{1k}r) = \overline{f(r)} \tag{24}$$

by Fourier-Bessel series expansion

$$R_{1k} + C_{1k} + \overline{C_{-1k}} = \frac{2}{J_2\big(\lambda_{1k}\big)^2}\int_0^1 rf(r)J_1\big(\lambda_{1k}r\big)\ dr$$

$$R_{-1k}+C_{-1k}+\overline{C_{1k}}=\frac{2}{J_2\big(\lambda_{1k}\big)^2}\int_0^1 r\overline{f(r)}J_1\big(\lambda_{1k}r\big)\ dr$$

$$u_t(0,r,\theta)=\sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}\big(r_{nk}R_{nk}e^{in\theta}+i\omega_{nk}C_{nk}e^{in\theta}-i\overline{\omega_{nk}}\overline{C_{nk}}e^{-in\theta}\big)J_n\Big(\frac{\lambda_{nk}}{R}r\Big)=0 \tag{25}$$

$$r_{nk}R_{nk}+i\omega_{nk}(C_{nk}+\overline{C_{-nk}})=0\,,\quad \forall n\in\mathbb{N}\,,k\in\mathbb{N} \tag{26}$$

$$r_{nk}R_{-nk}-i\overline{\omega_{nk}}(C_{-nk}+\overline{C_{nk}})=0\,,\quad \forall n\in\mathbb{N}\,,k\in\mathbb{N}\,. \tag{27}$$

In view of (21) it follows

$$\begin{aligned}R_{nk}&=-C_{nk}-\overline{C_{-nk}}\,,\quad \forall n\in\mathbb{N}\,,n\neq 1\,,k\in\mathbb{N}\\(i\omega_{nk}-r_{nk})(C_{nk}+\overline{C_{-nk}})&=0\,,\quad \forall n\in\mathbb{N}\,,k\in\mathbb{N}\\C_{nk}+\overline{C_{-nk}}=R_{nk}&=0\,,\quad \forall n\in\mathbb{N}\,,n\neq 1\,,k\in\mathbb{N}\end{aligned} \tag{28}$$

$$\begin{aligned}u(t,r,\theta)=&\sum_{k=1}^{\infty}\Big(R_{1k}e^{r_{1k}t}+(C_{1k}+\overline{C_{-1k}})e^{i\omega_{1k}t}\Big)J_1\Big(\frac{\lambda_{1k}}{R}r\Big)e^{i\theta}\\&+\sum_{k=1}^{\infty}\Big(\overline{R_{1k}}e^{r_{1k}t}+(\overline{C_{1k}}+C_{-1k})e^{-i\overline{\omega_{1k}}t}\Big)J_1\Big(\frac{\lambda_{1k}}{R}r\Big)e^{-i\theta}\end{aligned} \tag{29}$$

where

$$C_{1k}+\overline{C_{-1k}}=\frac{2}{J_2\big(\lambda_{1k}\big)^2}\int_0^1 rf(r)J_1\big(\lambda_{1k}r\big)\ dr-R_{1k}$$

$$R_{1k}=-\frac{2i\omega_{1k}}{J_2\big(\lambda_{1k}\big)^2(r_{1k}-i\omega_{1k})}\int_0^1 rf(r)J_1\big(\lambda_{1k}r\big)\ dr$$

$$C_{1k}+\overline{C_{-1k}}=\frac{2r_{1k}}{J_2\big(\lambda_{1k}\big)^2(r_{1k}-i\omega_{1k})}\int_0^1 rf(r)J_1\big(\lambda_{1k}r\big)\ dr\,.$$

Suppose that the membrane is fixed along the boundary circle $r=R$. The initial deflection $f(r)$ and the initial velocity depend only on r, not on θ, so that we expect that the vibration is radially symmetric. Hence the deflection $u=u(t,r)$ at any instant t and $u_{\theta\theta}=0$. So, in formula (6) we have only $n=0$ and the Bessel's equation (9) is only of order 0. Therefore, the expression (15) for the solution is brought to

$$u(t,r)=\sum_{k=1}^{\infty}(R_k e^{r_k t}+C_k e^{i\omega_k t}+\overline{C_k}e^{-i\overline{\omega_k}t})J_0\Big(\frac{\lambda_k}{R}r\Big)\,, \tag{30}$$

where J_0 denotes the Bessel function of order 0 and λ_k are the positive zeros of J_0. We note that

$$u_r(t,R)=\frac{1}{R}\sum_{k=1}^{\infty}\lambda_k(R_k e^{r_k t}+C_k e^{i\omega_k t}+\overline{C_k}e^{-i\overline{\omega_k}t})J_0'\big(\lambda_k\big)\,. \tag{31}$$

Now, by the definition (10) of Bessel functions it easily follows

$$\frac{dJ_0}{dx}(x)=-J_1(x)\,.$$

So, if we use the previous formula in (31), then we have

$$u_r(t,R)=-\frac{1}{R}\sum_{k=1}^{\infty}\lambda_k(R_k e^{r_k t}+C_k e^{i\omega_k t}+\overline{C_k}e^{-i\overline{\omega_k}t})J_1\big(\lambda_k\big)\,. \tag{32}$$

The following observability estimate holds true: If $T > 2R$ there exist two constants c_1 and c_2 such that

$$c_1 \sum_{k=1}^{\infty} \lambda_k^2 |J_1(\lambda_k)|^2 |C_k|^2 \le \int_0^T | \, u_r(t,R)|^2 dt \le c_2 \sum_{k=1}^{\infty} \lambda_k^2 |J_1(\lambda_k)|^2 |C_k|^2 \,.$$

This is an illustrative case of the general theory leading to simply the computations.

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