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EXAMPLES OF MORPHISMS OF OPERATOR EFFECT ALGEBRAS

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1. Basic definitions

Let $\mathscr{H}$ be a complex separable Hilbert space. The set $\mathcal{E}(\mathscr{H})$ of Hilbert space effects, i.e. bounded selfadjoint operators E such that $0 \leq (Ex, x) \leq (x, x)$ for all $x \in \mathscr{H}$ was a prototype of abstract effect algebra defined in [1].

Definition 1. A partial algebra $(E, \oplus, 0, 1)$ is called an *effect algebra if* $0, 1$ *are two distinguished elements and* $\oplus$ *is a partially defined binary operation on* E *satisfying* $\forall\, x, y.z \in E$

(E1) $x \oplus y = y \oplus x$ *if* $x \oplus y$ *is defined,*
(E2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ *if one side is defined,*
(E3) *For every* $x \in E$ *there exists a unique* $y \in E$ *such that* $x \oplus y = 1$*,*
(E4) *If* $1 \oplus x$ *is defined, then* $x = 0$*.*

A partial order on an effect algebra A is defined as

$$x \leq y \iff \exists z \in E \text{ for which } x \oplus z = y \text{ (we write } z = y \ominus x) . \tag{1}$$

Definition 2.

(i) Let $(E, \oplus, 0, 1)$ be an effect algebra. $\omega : E \to [0, 1] \subset \mathbb{R}$ is a *state if* $\omega(0) = 0$*,* $\omega(1) = 1$ *and if* $x \oplus y$ *is defined, then* $\omega(x \oplus y) = \omega(x) + \omega(y)$*.*
(ii) *A set* $\mathcal{M}$ *of states is called an ordering set of states if* $a \leq b \iff (\omega(a) \leq \omega(b)\ \forall \omega \in \mathcal{M})$*.*

It was proved in [2, Theorem 3] that if an effect algebra E has an ordering set of states, then E can be embedded into $\mathcal{E}(\mathscr{H})$ (i.e. there is an injective homomorphism $E \to \mathcal{E}(\mathscr{H})$).

One of the generalizations of effect algebra is (see, e.g., [2, 3])

Definition 3. A *generalized effect algebra* $(E, \oplus, 0)$ is a set E with element $0 \in E$ and partial binary operation $\oplus$ satisfying for any $x, y, z \in E$

(GE1) $x \oplus y = y \oplus x$ if one side is defined,
(GE2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,
(GE3) If $x \oplus y = x \oplus z$ then $y = z$,
(GE4) If $x \oplus y = 0$ then $x = y = 0$,
(GE5) $x \oplus 0 = x$ for all $x \in E$.

Here we shall investigate generalized effect algebras $\mathcal{G}_D(\mathscr{H})$ consisting of symmetric operators [4]. Let D be a dense linear subspace of $\mathscr{H}$ and

$$\mathcal{G}_D(\mathscr{H}) = \{A : D \to \mathscr{H}\,;\ A \text{ is a positive linear operator defined on } D\} . \tag{2}$$

Obviously, $\mathcal{G}_D(\mathscr{H})$ with usual sum of operators is a generalized effect algebra.

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2. Morphisms of operator generalized effect algebras

First, we recall the following result [5], [7, Chap. 2.7].

Theorem 4. *Let $\varphi : \mathcal{E}(\mathscr{H}) \to \mathcal{E}(\mathscr{H})$ be a bijective map which preserves the order in both directions and let there exist $\lambda \in (0,1)$ such that $\varphi(\lambda I) = \lambda I$. Then there exists a unitary or antiunitary operator U such that*

$$\varphi(A) = UAU^* \qquad (A \in \mathcal{E}(\mathscr{H}))\,. \tag{3}$$

Note that since the order is given by (1), any bijective automorphism of $\mathcal{E}(\mathscr{H})$ preserves the order in both direction.

It is now natural to ask if Theorem 4 can be generalized to generalized effect algebras.

We may identify the bounded operators in $\mathcal{G}_D(\mathscr{H})$ with their extensions to the whole $\mathscr{H}$. Let D be a fixed dense subspace of $\mathscr{H}$. If $\varphi : \mathcal{G}_D(\mathscr{H}) \to \mathcal{G}_D(\mathscr{H})$ is given by (3), then $U^*D \subset D$, otherwise UAU^* need not be defined for unbounded operators $A \in \mathcal{G}_D(\mathscr{H})$.

We are able to give only trivial examples of homomorphisms of two operator generalized effect algebras. First we consider the case of the "smallest" dense subspaces D_1, D_2.

Example 5. Let $\{e_k\}_{k=0}^{\infty}$, $\{f_k\}_{k=0}^{\infty}$ be orthonormal bases of $\mathscr{H}$. Let D_1 and D_2 be the linear (not closed) spans of $\{e_k\}$ and $\{f_k\}$, respectively. Then

$$Ue_k = f_k\,, \quad U^*f_k = e_k\,, \qquad k = 0,1,2\ldots$$

defines a unitary operator for which

$$\varphi(A) = UAU^* \qquad (A \in \mathcal{G}_{D_1}(\mathscr{H}))$$

is a bijective homomorphism $\mathcal{G}_{D_1}(\mathscr{H}) \to \mathcal{G}_{D_2}(\mathscr{H})$.

If $D_2 \subset D_1$, then the restriction is a natural example of homomorphisms $\mathcal{G}_{D_1}(\mathscr{H}) \to \mathcal{G}_{D_2}(\mathscr{H})$. Since there exists at most one extension of $A \in \mathcal{G}_{D_2}(\mathscr{H})$ to the bigger space D_2 we obtain the following example. This is also of the form (3) with $U = I$, the identity operator.

Example 6. Let $D_2 \subset D_1$ be dense subsets of $\mathscr{H}$. Then

$$\varphi(A) = A\big|D_2 \qquad (A \in \mathcal{G}_{D_1}(\mathscr{H}))$$

is an injective but not surjective homomorphism $\mathcal{G}_{D_1}(\mathscr{H}) \to \mathcal{G}_{D_2}(\mathscr{H})$.

We conclude with

Conjecture. Let $\mathscr{H}$ be a complex separable Hilbert space. There exist two dense subspaces D_1, D_2 such that there are no nonzero homomorphisms

$$\varphi_1 : \mathcal{G}_{D_1}(\mathscr{H}) \to \mathcal{G}_{D_2}(\mathscr{H}) \quad \text{and} \quad \varphi_2 : \mathcal{G}_{D_2}(\mathscr{H}) \to \mathcal{G}_{D_1}(\mathscr{H}).$$

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