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D–WEAK OPERATOR TOPOLOGY AND SOME ITS PROPERTIES

MARCEL POLAKOVIČ

1. Introduction

Effect algebras as algebraic structures were introduced by Foulis and Bennett in [1]. Generalized effect algebras are natural generalization of effect algebras (without a top element 1). In [2], an example $\mathcal{G}_D(\mathcal{H})$ of generalized effect algebra was investigated, which consisted from positive operators defined on a dense subspace D of a Hilbert space $\mathcal{H}$. In [3], some topological properties of $\mathcal{G}_D(\mathcal{H})$ were investigated, where the D–weak operator topology on $\mathcal{G}_D(\mathcal{H})$ was introduced. In the present contribution, some more properties of this topology are mentioned.

2. Preliminaries and results

Let us define the generalized effect algebra.

Definition 1.

(1) A *generalized effect algebra* $(E, \oplus, 0)$ is a set E with element $0 \in E$ and partial binary operation $\oplus$ satisfying for any $x, y, z \in E$ conditions
  - (GE1) $x \oplus y = y \oplus x$ if one side is defined
  - (GE2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined
  - (GE3) If $x \oplus y = x \oplus z$ then $y = z$
  - (GE4) If $x \oplus y = 0$ then $x = y = 0$
  - (GE5) $x \oplus 0 = x$ for all $x \in E$.

(2) Define a binary relation $\leq$ on E by
  $x \leq y$ iff for some $z \in E$, $x \oplus z = y$.

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space and let D be its dense linear subspace. Let

$$\mathcal{G}_D(\mathcal{H}) = \{A : D \to \mathcal{H} \mid A \text{ is a positive linear operator defined on } D\}.$$

In [2], it is shown that $(\mathcal{G}_D(\mathcal{H}); \oplus, 0)$ is a generalized effect algebra where 0 is the null operator and $\oplus$ is the usual sum of operators defined on D.

Let $Q \in \mathcal{G}_D(\mathcal{H})$. The set

$$[0, Q]_{\mathcal{G}_D(\mathcal{H})} = \{A \in \mathcal{G}_D(\mathcal{H}) \mid 0 \leq A \leq Q\}$$

is called an interval in $\mathcal{G}_D(\mathcal{H})$ and it has a natural structure of effect algebra (see [4]) and consequently, also of generalized effect algebra.

D-weak operator topology on $\mathcal{G}_D(\mathcal{H})$ was defined in [3] as the weakest topology such that all functions $f_x : \mathcal{G}_D(\mathcal{H}) \to \mathbb{R}$ where $f_x(A) = (x, Ax), x \in D$, are continuous. We denote this topology by $\tau^w_{D,\mathcal{G}}$. In [3] it is denoted by τ_D. Let us note that $\tau^w_{D,\mathcal{G}}$ and τ_D are strictly not completely the same as in [3] the set $\mathcal{G}_D(\mathcal{H})$ is defined in slightly different manner than it is in the present contribution.

Let $\mathcal{S}_D(\mathcal{H})$ denotes the set of all symmetric linear operators on $\mathcal{H}$ with domain D. Clearly

$$\mathcal{G}_D(\mathcal{H}) = \{A \in \mathcal{S}_D(\mathcal{H}) \mid A \geq 0\}.$$

$\mathcal{S}_D(\mathcal{H})$ is a linear space and for $x, y \in D$ the function $\rho_{x,y} : \mathcal{S}_D(\mathcal{H}) \to \mathbb{R}$, $\rho_{x,y}(A) = |(x, Ay)|$ is a seminorm. The family of seminorms $\{\rho_{x,y}\}_{x,y \in D}$ separates points. So $(\mathcal{S}_D(\mathcal{H}), \{\rho_{x,y}\}_{x,y\in D})$ is a locally convex space. The corresponding natural topology we denote by $\tau^w_{D,\mathcal{S}}$ and call the

D-weak operator topology on $\mathcal{S}_D(\mathcal{H})$. It can be seen that $\tau^w_{D,\mathcal{S}}$ is a natural generalization of the usual weak operator topology on $\mathcal{B}(\mathcal{H})$ (the set of all bounded operators defined on the whole space $\mathcal{H}$). (The weak operator topology on $\mathcal{B}(\mathcal{H})$ is the natural topology of the locally convex space $(\mathcal{B}(\mathcal{H}), \{\rho_{x,y}\}_{x,y\in\mathcal{H}})$, here $\rho_{x,y}(A) = |(x, Ay)|$ for $A \in \mathcal{B}(\mathcal{H})$, $x, y \in \mathcal{H}$.)

If $A_\alpha, A \in \mathcal{S}_D(\mathcal{H})$, A_α is a net, then $A_\alpha \overset{\tau^w_{D,\mathcal{S}}}{\longrightarrow} A$ if and only if $\big(x, (A_\alpha - A)y\big) \to 0$ for all $x, y \in D$. But from the polarization formula we have that this happens if and only if $(x, (A_\alpha - A)x) \to 0$ for all $x \in D$, i.e. $(x, A_\alpha x) \to (x, Ax)$ for all $x \in D$. So the topology $\tau^w_{D,\mathcal{S}}$ on $\mathcal{S}_D(\mathcal{H})$ is generated by the family of seminorms $\{\rho_{x,x}\}_{x\in D}$ (here $\rho_{x,x}(A) = |(x, Ax)|, x \in D, A \in \mathcal{S}_D(\mathcal{H})$) which separates points.

Now let us note that for $A \in \mathcal{G}_D(\mathcal{H})$ we have $f_x(A) = (x, Ax) = |(x, Ax)| = \rho_{x,x}(A)$ as $A \geq 0$. So the D-weak operator topology $\tau^w_{D,\mathcal{G}}$ on $\mathcal{G}_D(\mathcal{H})$ is generated by the same family of functions $\{\rho_{x,x}\}_{x\in D}$ as the D-weak operator topology $\tau^w_{D,\mathcal{S}}$ on $\mathcal{S}_D(\mathcal{H})$. More precisely, the D-weak operator topology on $\mathcal{G}_D(\mathcal{H})$ is generated by the family of functions $\{\rho'_{x,x}\}_{x\in D}$ where $\rho'_{x,x}$ is the restriction of $\rho_{x,x}$ to the set $\mathcal{G}_D(\mathcal{H})$. (Let us recall that $\rho_{x,x}$ is defined on $\mathcal{S}_D(\mathcal{H})$.)

Now it can easily be shown that the D-weak operator topology $\tau^w_{D,\mathcal{G}}$ on $\mathcal{G}_D(\mathcal{H})$ is the corresponding relative topology – $\tau^w_{D,\mathcal{S}}$ restricted to $\mathcal{G}_D(\mathcal{H})$ ($\tau^w_{D,\mathcal{G}} = \tau^w_{D,\mathcal{S}} \cap \mathcal{G}_D(\mathcal{H})$).

Let us characterize the topology $\tau^w_{D,\mathcal{G}}$ in terms of convergence. Let $A_\alpha, A \in \mathcal{G}_D(\mathcal{H})$, A_α is a net, then $A_\alpha@ \overset{\tau^w_{D,\mathcal{S}}}{\longrightarrow} A$ if and only if $(x, (A_\alpha - A)x) \to 0$ for all $x \in D$, i.e. $(x, A_\alpha x) \to (x, Ax)$ for all $x \in D$.

Let us denote by $\mathcal{B}^+_D(\mathcal{H})$ the set of all bounded positive linear operators defined on the dense subspace D of $\mathcal{H}$. So

$$B^+_D(\mathcal{H}) = \{A : D \to \mathcal{H} \text{ linear} \mid A \geq 0, \exists M \in \mathbb{R} : ||Ax|| \leq M||x|| \text{ for } x \in D\}.$$

Clearly $\mathcal{B}^+_D(\mathcal{H}) \subseteq \mathcal{G}_D(\mathcal{H})$.

Now, we shall mention some results.

Theorem 2. (i) *The closure of the set* $\mathcal{B}^+_D(\mathcal{H})$ *in the topology* $\tau^w_{D,\mathcal{G}}$ *is* $\mathcal{G}_D(\mathcal{H})$.
(ii) *The closure of the set* $\mathcal{G}_D(\mathcal{H}) \setminus \mathcal{B}^+_D(\mathcal{H})$ *in the topology* $\tau^w_{D,\mathcal{G}}$ *is* $\mathcal{G}_D(\mathcal{H})$.

So bounded operators and unbounded operators are dense subsets of $\mathcal{G}_D(\mathcal{H})$ in D-weak operator topology on $\mathcal{G}_D(\mathcal{H})$ or, equivalently, the closure of each of these sets (in D-weak operator topology on $\mathcal{G}_D(\mathcal{H})$) is the whole set $\mathcal{G}_D(\mathcal{H})$. Now we show that each interval $[0, Q]_{\mathcal{G}_D(\mathcal{H})}$, where $Q \in \mathcal{G}_D(\mathcal{H})$ is an unbounded operator, has a similar property. Namely if we take the set of all bounded operators in $[0, Q]_{\mathcal{G}_D(\mathcal{H})}$ then its closure in D-weak operator topology on $\mathcal{G}_D(\mathcal{H})$ is just the whole set $[0, Q]_{\mathcal{G}_D(\mathcal{H})}$. The same is true also for the set of all unbounded operators in $[0, Q]_{\mathcal{G}_D(\mathcal{H})}$ (which is the set $[0, Q]_{\mathcal{G}_D(\mathcal{H})} \setminus \mathcal{B}^+_D(\mathcal{H})$).

Theorem 3. *Let* $Q \in \mathcal{G}_D(\mathcal{H})$ *be an unbounded operator.*
(i) *The closure of the set* $\mathcal{B}^+_D(\mathcal{H}) \cap [0, Q]_{\mathcal{G}_D(\mathcal{H})}$ *in the topology* $\tau^w_{D,\mathcal{G}}$ *is* $[0, Q]_{\mathcal{G}_D(\mathcal{H})}$.
(ii) *The closure of the set* $[0, Q]_{\mathcal{G}_D(\mathcal{H})} \setminus \mathcal{B}^+_D(\mathcal{H})$ *in the topology* $\tau^w_{D,\mathcal{G}}$ *is* $[0, Q]_{\mathcal{G}_D(\mathcal{H})}$.

References

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Institute of Computer Science and Mathematics, Department of Mathematics, Slovak University of Technology, Faculty of Electrical Engineering and Information Technology, Bratislava, Slovakia

E-mail address: marcel.polakovic@stuba.sk