10th Workshop Functional Analysis and its Applications, September 7–12, 2015, Kočovce, Slovakia, 18–23

# REGULARIZED OPTIMAL CONTROL PROBLEM FOR AN ANISOTROPIC PLATE VIBRATING AGAINST AN ELASTIC FOUNDATION

M. KEČKEMÉTYOVÁ, I. BOCK

Abstract. We deal with an optimal control problem governed by a nonlinear hyperbolic initial-boundary value problem describing the perpendicular vibrations of an anisotropic plate against an elastic foundation. A variable thickness of a plate plays the role of a control variable. The original equation for the deflection is regularized in order to derive necessary optimality conditions.

## 1. Introduction

Shape design optimization problems belong to frequently solved problems with many engineering applications. We deal here with an optimal design problem for an elastic anisotropic plate vibrating against an elastic called also the Winkler foundation. A variable thickness of a plate plays the role of a control variable. The similar problems for the axisymmetric plate and the stationary elastic Bernoulli beam are investigated in [5] and [6] respectively. We have considered the control problem for an elastic beam vibrating against an elastic foundation in [2]. Due to the variable thickness $e$ the equation for the movement of the plate has the form

$$e(x)u_{tt} + (e^3(x)a_{ijk\ell}u_{x_ix_j})_{x_kx_\ell} + q(x)[u + \tfrac{1}{2}(e(x) - e_{max})]^+ = f(t,x) \text{ in } (0,T] \times \Omega\,.$$

In order to derive not only the existence of an optimal variable thickness but also the necessary optimality conditions we regularize the function $\omega \mapsto \omega^+$ by

$$\omega \mapsto g_\delta(\omega),\ g_\delta(\omega) = \begin{cases} 0 & \text{for } w \le 0\,, \\ \frac{6}{\delta^2}\omega^3 - \frac{8}{\delta^3}\omega^4 + \frac{3}{\delta^4}\omega^5 & \text{for } 0 < \omega < \delta\,, \\ \omega & \text{for } \omega \ge \delta\,. \end{cases}$$

We remark that instead of the function $g_\delta$ we can use any not negative nondecreasing function $g \in C^2(\mathbb{R})$ of the variable $\omega$ vanishing for $\omega \le 0$, equaled to $\omega$ for $\omega \ge \delta$ and fulfilling $\max_{\omega\in[0,\delta]} |g_\delta(\omega)| \le M\delta$.

## 2. Solving of the state problem

2.1. **Setting of the state problem.** We consider an anisotropic plate with the distance $\frac{1}{2}e_{max}$ between the middle surface and the foundation. The middle surface $\Omega \subset \mathbb{R}^2$ is assumed to have a Lipschitz continuous boundary $\partial\Omega$ with a unit outer normal vector $\vec{n}(\xi),\ \xi \in \partial\Omega$. The variable thickness of the plate is expressed by a positive function $x \mapsto e(x),\ x \in \bar{\Omega}$, the positive constant $\rho$ is the density of the material and a positive function $x \mapsto q_0(x),\ x \in \bar{\Omega}$ represents the stiffness of the foundation. The plate is clamped on its boundary. Let $F : (0,T] \times \Omega \mapsto \mathbb{R}$ be a perpendicular load per a square unit acting on the plate. Then the vertical displacement $u : (0,T] \times \Omega \mapsto \mathbb{R}$ is a solution of the following hyperbolic equation

$$\tfrac{1}{2}\rho e(x)u_{tt} + (e^3(x)A_{ijk\ell}u_{x_ix_j})_{x_kx_\ell} + q_0(x)g_\delta(u + \tfrac{1}{2}(e(x) - e_{max})) = F(t,x) \text{ in } (0,T] \times \Omega$$

---

2010 *Mathematics Subject Classification.* 35L70, 49J20, 74K20.

*Key words and phrases.* Anisotropic plate, elastic foundation, optimal control, variable thickness.

This work was financially supported by the Ministry of Education of Slovak Republic under grant VEGA-1/0426/12 and by grant of Science and Technology Assistance Agency no. APVV-0246-12.

with a symmetric and positively definite tensor $\{A_{ijk\ell}\}$. Let $u_0,\ v_0 : \Omega \mapsto \mathbb{R}$ be the initial displacement and velocity, $a_{ijk\ell} = \frac{2}{\rho} A_{ijk\ell}$, $q = \frac{2q_0}{\rho}$, $f = \frac{2F}{\rho}$ be the new mechanical and material characteristics. Then the vertical displacement $u : (0,T] \times \Omega \mapsto \mathbb{R}$ solves the hyperbolic initial-boundary value problem

$$e(x)u_{tt} + (e^3(x)a_{ijk\ell}u_{x_ix_j})_{x_kx_\ell} + q(x)g_\delta(u + \tfrac{1}{2}(e(x) - e_{max})) = f(t,x) \text{ in } (0,T] \times \Omega, \tag{1}$$

$$u(t,\xi) = \frac{\partial u}{\partial \vec{n}}(t,\xi) = 0,\ t \in (0,T],\ \xi \in \partial\Omega \tag{2}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = v_0(x),\ x \in \Omega. \tag{3}$$

We introduce the Hilbert spaces

$$H \equiv L_2(\Omega),\ H^k(\Omega) = \{y \in H:\ D^\alpha y \in H,\ |\alpha| = k\},$$

with the standard inner products $(\cdot,\cdot),\ (\cdot,\cdot)_k$, the norms $|\cdot|_0,\ \|\cdot\|_k$ and

$$V \equiv \mathring{H}^2(\Omega) = \{y \in H^2(\Omega):\ y(\xi) = \frac{\partial y}{\partial \vec{n}}(\xi) = 0,\ \xi \in \partial\Omega \text{ (in the sense of traces)}\}$$

with the inner product and the norm

$$((y,z)) = \int_\Omega y_{x_ix_j}(x) z_{x_ix_j}(x)\,dx,\ \|y\| = ((y,y))^{1/2},\ y,\ z \in V.$$

We denote by $V^*$ the dual space of linear bounded functionals over $V$ with duality pairing $\langle F, y\rangle_* = F(y),\ F \in V^*,\ y \in V$. It is a Banach space with a norm $\|\cdot\|_*$.
The spaces $V,\ H,\ V^*$ form the Gelfand triple meaning the dense and compact embedings

$$V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V^*.$$

We set $I = (0,T),\ Q = I \times \Omega$. For a Banach space $X$ we denote by $L_p(I;X)$ the Banach space of all functions $y : I \mapsto X$ such that $\|y(\cdot)\|_X \in L_p(0,T),\ p \geq 1$, by $L_\infty(I;X)$ the space of essentially bounded functions with values in $X$, by $C(\bar{I};X)$ the space of continuous functions $y : \bar{I} \mapsto X,\ \bar{I} = [0,T]$. For $k \in N$ we denote by $C^k(\bar{I};X)$ the spaces of $k$-times continuously differentiable functions defined on $\bar{I}$ with values in $X$. If $X$ is a Hilbert space we set

$$H^k(I;X) = \{v \in C^{k-1}(\bar{I};X):\ \frac{d^k v}{dt^k} \in L_2(I;X)\}$$

the Hilbert spaces with the inner products

$$(u,v)_{H^k(I,X)} = \int_I [(u,v)_X + \sum_{j=1}^k (u^j, v^j)_X]\,dt,\ k \in N.$$

We denote by $\dot{w},\ \ddot{w}$ and $\dddot{w}$ the first, the second and the third time derivative of a function $w : I \to X$. In order to derive necessary optimality conditions in the next chapter we assume stronger regularity of data:

$$\begin{aligned} &u_0 \in V \cap H^4(\Omega),\ u_0(x) + \tfrac{1}{2}e(x) \leq \tfrac{1}{2}e_{max}\,\forall x \in \Omega;\\ &v_0 \in V,\ f \in H^1(I;H),\ 0 < q \in C(\bar{\Omega}),\ e \in E_{ad};\\ &E_{ad} = \Big\{e \in H^2(\Omega):\ 0 < e_{\min} \leq e(x) \leq e_{\max}\ \forall x \in \bar{\Omega},\ \|e\|_2 \leq \hat{e}\Big\}. \end{aligned} \tag{4}$$

The symmetric and positively definite fourth-order tensor $a_{ijk\ell}$ fulfils $\forall\ i,j,k,\ell \in \{1,2\}$:

$$a_{ijk\ell} = a_{k\ell ij} = a_{jik\ell}, \quad \alpha\,\varepsilon_{ij}\varepsilon_{ij} \leq a_{ijk\ell}\,\varepsilon_{ij}\varepsilon_{k\ell} \leq \beta\,\varepsilon_{ij}\varepsilon_{ij}\ \forall\ \{\varepsilon_{ij}\} \in \mathbb{R}^{2\times2}_{sym},\ \alpha > 0,$$

where the Einstein summation convention is employed and $\mathbb{R}^{2\times2}_{sym}$ is the set of all second-order symmetric tensors.

**Definition 1.** A function $u$ is a weak solution of the problem (1)-(3) if $\ddot{u} \in L_2(Q), u \in L_2(I;V)$, there hold the identity

$$\begin{aligned}&\int_Q \left[e(x)\ddot{u}y + e^3(x)a_{ijk\ell}u_{x_ix_j}y_{x_kx_\ell} + q(x)g_\delta(u + \tfrac{1}{2}(e(x) - e_{max}))y\right]\,dx\,dt \\ &= \int_Q f(t,x)y\,dx\,dt \;\forall\, y \in L_2(I;V)\end{aligned} \tag{5}$$

and the initial conditions

$$u(0) = u_0,\ \dot{u}(0) = v_0. \tag{6}$$

2.2. **Existence and uniqueness of the state problem.** We verify the existence and uniqueness of a weak solution.

**Theorem 2.** *There exists a unique solution $u$ of the problem (5),(6) such that $\dot{u} \in L_\infty(I;V) \cap C(\bar{I};H^{2-\varepsilon}(\Omega))$ $\forall \varepsilon > 0$, $\ddot{u} \in L_\infty(I;H)$, $\dddot{u} \in L_2(I;V^*)$ and there holds the estimate*

$$\|\dot{u}\|_{L_\infty(I,V)} + \|\ddot{u}\|_{L_\infty(I,H)} + \|\dddot{u}\|_{L_2(I,V^*)} \le C_0(\alpha,\beta,e_{\min},e_{\max},\hat{e},u_0,v_0,f,q). \tag{7}$$

*Proof.* Using the Galerkin method we obtain the approximation $u_m$ of a solution (5),(6) which can be extended to the whole interval $[0,T]$ with the a priori estimates

$$\|\dot{u}_m\|^2_{C(\bar{I},H)} + \|u_m\|^2_{C(\bar{I},V)} \le C_1(\alpha,\beta,e_{\min},e_{\max},\hat{e},u_0,v_0,f,q), \tag{8}$$

$$\|\ddot{u}_m\|^2_{C(\bar{I},H)} + \|\dot{u}_m\|^2_{C(\bar{I},V)} \le C_2(\alpha,\beta,e_{\min},e_{\max},\hat{e},u_0,v_0,f,q). \tag{9}$$

Applying the estimates (8), (9), the Aubin-Lions compact imbedding theorem [4], Sobolev imbedding theorems and the interpolation theorems in Sobolev spaces [3] we obtain for a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$) a function $u \in C(\bar{I},V)$ with $\dot{u} \in L_\infty(I,V),\ \ddot{u} \in L_\infty(I,H)$ and the convergences

$$\begin{aligned}\ddot{u}_m &\rightharpoonup^* \ddot{u} && \text{in } L_\infty(I,H),\\ \dot{u}_m &\rightharpoonup^* \dot{u} && \text{in } L_\infty(I;V),\\ u_m &\to u && \text{in } C(\bar{I};V),\\ u_m &\to u && \text{in } C^1(\bar{I};H^{2-\varepsilon}(\Omega))\ \forall\,\varepsilon > 0,\\ u_m &\to u && \text{in } C^1(\bar{I};C(\bar{\Omega})).\end{aligned} \tag{10}$$

The convergence process (10) implies that a function $u$ fulfils for a.e. $t \in I$

$$\begin{aligned}&\int_\Omega \left[e\ddot{u}w + e^3(x)a_{ijk\ell}u_{x_ix_j}w_{x_kx_\ell} + q(x)g_\delta(u + \tfrac{1}{2}(e(x) - e_{\max}))w\right]\,dx \\ &= \int_\Omega fw\,dx,\ \forall\, w \in V.\end{aligned} \tag{11}$$

The identity (5) follows directly after setting $w \equiv y(t,\cdot),\ y \in L_2(I;V)$ in (11).

Due to the differentiability of $g_\delta,\ f$ and the relation $\dot{u} \in L_\infty(I;V)$ we obtain the third time derivative $\dddot{u} \in L_2(I;V^*)$ fulfilling

$$\begin{aligned}&\int_Q \left[e(x)\dddot{u}y + e^3(x)a_{ijk\ell}(\dot{u}_{x_ix_j}y_{x_kx_\ell} + q(x)g'_\delta(u + \tfrac{1}{2}(e(x) - e_{max}))\dot{u}\,y\right]\,dx\,dt \\ &= \int_Q \dot{f}(t,x)y\,dx\,dt \;\forall\, y \in L_2(I;V).\end{aligned} \tag{12}$$

The estimate (9) together with the convergences (10) and the relation (12) implies the estimate (7). The proof of the uniqueness can be performed in a standard way using the Gronwall lemma.

**Remark 3.** The constant $C_0(\alpha,\beta,e_{\min},e_{\max},\hat{e},u_0,v_0,f,q)$ in the estimate (7) does not depend on $\delta$ for $\delta \in (0,\delta_0)$.

## 3. Optimal control problem

3.1. **The existence of an optimal thickness.** We consider a cost functional $J : L_2(I;V) \times H^2(\Omega) \mapsto \mathbb{R}$ fulfilling the assumption

$$u_n \rightharpoonup u \text{ in } L_2(I;V),\ e_n \rightharpoonup e \text{ in } H^2(\Omega) \Rightarrow J(u,e) \le \liminf_{n\to\infty} J(u_n, e_n) \tag{13}$$

and formulate

*Optimal control problem* $\mathcal{P}$ : To find a control $e_* \in E_{ad}$ such that

$$J(u(e_*), e_*) \le J(u(e), e)\ \forall e \in E_{ad}, \tag{14}$$

where $u(e)$ is a (unique) weak solution of the Problem (1)-(3).

**Theorem 4.** *There exists a solution of the Optimal control problem* $\mathcal{P}$*.*

*Proof.* We use the weak lower semicontinuity property of the functional $J$ and the compactness of the admissible set $E_{ad}$ of thicknesses in the space $C(\bar{\Omega})$. Let $\{e_n\} \subset E_{ad}$ be a minimizing sequence for (14). The set $E_{ad}$ is convex and closed and hence a weakly closed in $H^2(\Omega)$ as the closed convex set. Then there exists a subsequence of $\{e_n\}$ (denoted again by $\{e_n\}$) and an element $e_* \in E_{ad}$ such that

$$e_n \rightharpoonup e_* \text{ in } H^2(\Omega),\ e_n \to e_* \text{ in } C(\bar{\Omega}). \tag{15}$$

The *a priori* estimates (7), Sobolev imbedding theorems and the Ascoli theorem on uniform convergence on $\bar{I}$ imply the existence of a function $u^* \in C(\bar{I};V)$ such that $\dot{u} \in L_\infty(I;V) \cap C(\bar{I};H)$, $\ddot{u} \in L_\infty(I;H)$ and the convergences

$$\begin{aligned} &\ddot{u}(e_n) \rightharpoonup^* \ddot{u}^* \text{ in } L_\infty(I;H),\\ &\dot{u}(e_n) \rightharpoonup^* \dot{u}^* \text{ in } L_\infty(I;V),\ \dot{u}(e_n) \to \dot{u}^* \text{ in } C(\bar{I};H),\\ &u(e_n) \rightharpoonup^* u^* \text{ in } L_\infty(I;V),\ u(e_n) \to u^* \text{ in } C(\bar{I};C(\bar{\Omega})) \end{aligned} \tag{16}$$

for a chosen subsequence. Functions $u_n \equiv u(e_n)$ solve the initial value state problem (5),(6) for $e \equiv e_n$. Using a uniform Lipschitz continuity of $g_\delta$, and the convergences (15),(16) we obtain that $u^*$ solves the problem (5),(6).

We have then $u^* \equiv u(e_*)$ due to Theorem 2.2 and hence

$$u(e_n) \rightharpoonup u(e_*) \text{ in } L_2(I;V), \quad e_n \rightharpoonup e \text{ in } H^2(\Omega).$$

Property (13) then imply that $u(e_*)$ is a minimum of a functional $J$.

3.2. **Necessary optimality conditions.** Let us introduce the Banach space

$$\mathcal{W} = \{w \in L_2(I;V) :\ \dot{w} \in L_2(Q),\ \ddot{w} \in L_2(I;V^*)\}.$$

with a norm

$$\|w\|_{\mathcal{W}} = \|w\|_{L_2(I;V)} + \|\dot{w}\|_{L_2(Q)} + \|\ddot{w}\|_{L_2(I;V^*)}.$$

In a similar way as in [1], [2] the following theorem about Fréchet differentiability of the mapping $e \mapsto u(e)$ can be verified.

**Theorem 5.** *The mapping* $u(\cdot) : E_{ad} \to \mathcal{W}$ *is Fréchet differentiable and its derivative* $z \equiv z(h) = u'(e)h \in \mathcal{W},\ h \in H^2(\Omega)$ *fulfils for every* $e \in E_{ad}$ *uniquely the problem*

$$\mathcal{A}(e)z = -\mathcal{B}(e)h,\ z(0) = \dot{z}(0) = 0 \tag{17}$$

*with the operators* $\mathcal{A}(e):\mathcal{W}\to L_2(I;V^*)$, $\mathcal{B}(e):H^2(\Omega)\to L_2(I;V^*)$ *defined by*

$$\langle\langle\mathcal{A}(e)z,y\rangle\rangle=\int_0^T\langle\ddot{z},ey\rangle\,dt+\int_Q\left[e^3a_{ijk\ell}z_{x_ix_j}y_{x_kx_\ell}+q(x)g_\delta'(\omega(e))zy\right]\,dx\,dt, \tag{18}$$

$$\langle\langle\mathcal{B}(e)h,y\rangle\rangle=\int_Q h[\ddot{u}(e)y+3e^2a_{ijk\ell}u_{x_ix_j}(e)y_{x_kx_\ell}+\frac{1}{2}q(x)g_\delta'(\omega(e))y]\,dx\,dt, \tag{19}$$

$$\omega(e)=u(e)+\tfrac{1}{2}(e-e_{\max}),\ y\in L_2(I;V).$$

In order to derive necessary optimality conditions we assume that the cost functional $J(\cdot,\cdot):L_2(I;V)\times H^2(\Omega)\to\mathbb{R}$ is Fréchet differentiable.

The optimal control problem can be expressed in a form

$$j(e_*)=\min_{e\in E_{ad}}j(e),\ j(e)=J(u(e),e). \tag{20}$$

The functional $j$ in (20) is Fréchet differentiable and its derivative in $e_*\in E_{ad}$ has the form

$$\langle j'(e_*),h\rangle=\langle\langle J_u(u(e_*),e_*),u'(e_*)h\rangle\rangle+\langle J_e(u(e_*),e_*),h\rangle_{-2},\ h\in H^2(\Omega) \tag{21}$$

with the duality pairings $\langle\langle\cdot,\cdot\rangle\rangle$, $\langle\cdot,\cdot\rangle_{-2}$ between $L_2(I;V)^*$ and $L_2(I;V)$, $(H^2(\Omega))^*$ and $H^2(\Omega)$ respectively.

The optimal thickness $e_*\in E_{ad}$ fulfils the variational inequality

$$\langle j'(e_*),e-e_*\rangle_{-2}\geq 0\ \forall e\in E_{ad}. \tag{22}$$

which can be expressed in a form

$$\langle\langle J_u(u(e_*),e_*),u'(e_*)(e-e_*)\rangle\rangle+\langle J_e(u(e_*),e_*),e-e_*\rangle_{-2}\geq 0\ \forall e\in E_{ad}. \tag{23}$$

Applying Theorem 3.2 we obtain necessary optimality conditions in a form of a system with an adjoint state $p^*$:

**Theorem 6.** *The optimal thickness* $e_*$, *the corresponding state (deflection)* $u^*\equiv u(e_*)$ *and the adjoint state* $p^*\equiv p(e_*)$ *are solutions of the initial value problem*

$$\int_Q\left[e_*u_{tt}^*y+e_*^3(x)a_{ijk\ell}u^*_{x_ix_j}y_{x_kx_\ell}+q(x)g_\delta(u^*)y\right]\,dx\,dt=\int_Q f(t,x)y\,dx\,dt\ \forall\,y\in L_2(I;V),$$
$$u^*(0)=u_0,\ u_t^*(0)=v_0,$$
$$\mathcal{A}(e_*)p^*=-J_u(u^*,e_*);\ p^*(T)=p_t^*(T)=0,$$
$$\langle\langle\mathcal{B}(e_*)(e-e_*),p^*\rangle\rangle+\langle J_e(u^*,e_*),e-e_*\rangle_{-2}\geq 0\ \forall e\in E_{ad}.$$

**Remark 7.** If the partial derivative $e\mapsto J_e(u(e),e)$ is strongly monotone i.e.

$$\langle J_e(u(e_1),e_1)-J_e(u(e_2),e_2),e_1-e_2\rangle_2\geq N\|e\|_2^2\ \forall e_1,\,e_2\in H^2(\Omega),\ N>0,$$

then it is possible after using the variational inequality (23) to obtain for sufficiently large $N$ the uniqueness of the Optimal control $e_*$.

**Remark 8.** Let $\delta_n$ be the sequence of positive numbers fulfilling $\lim_{n\to\infty}\delta_n=0$. Using the same approach as in the proofs of Theorems 2.2 and 3.2 we can derive the convergence of the sequence of solutions $\{u_{\delta_n}\}$ of the state problem (5), (6) with $\delta\equiv\delta_n$ to a solution of the original initial-boundary value problem for a beam vibrating against Winkler foundation with a function $u\mapsto[u+\frac{1}{2}(e-e_{max})]^+$ instead of a regularized $u\mapsto g_\delta(u+\frac{1}{2}(e-e_{max}))$. It is an open and interesting question to investigate the corresponding sequences of optimal controls and of necessary optimality conditions.

## References


[1] Bock, I. and Lovíšek, J.: Optimal control of a viscoelastic plate bending respect to a thickness. *Math. Nachrichten* **125** (1986), 135–151.

[2] Bock, I. and Kečkemétyová, M.: Regularized optimal control problem for a beam vibrating against an elastic foundation. *Tatra Mountains Math. Publ.* **63** (2015), to appear.

[3] Eck C., Jarušek, J. and Krbec, M.: *Unilateral Contact Problems in Mechanics. Variational Methods and Existence Theorems.* Monographs & Textbooks in Pure & Appl. Math. No. 270 (ISBN 1-57444-629-0). Chapman & Hall/CRC (Taylor & Francis Group), Boca Raton – London – New York – Singapore 2005.

[4] Lions, J.L.: *Quelques méthodes de résolutions des problémes aux limites non linéaires.* Dunod, Paris, 1969.

[5] Salač, P.: Shape optimization of elastic axisymmetric plate on an elastic foundation. *Applications of Mathematics* 40 (1995), 319-338.

[6] Šimeček, R.: Optimal design of an elastic beam with a unilateral elastic foundation: Semicoercive case. *Applications of Mathematics* **58**, No.3 (2013), 329–346.


Institute of Computer Science and Mathematics, FEI Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava 1, Slovak Republic

*E-mail address*: igor.bock@stuba.sk, maria.keckemetyova@stuba.sk