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# ON THE RELATIVE NUMERICAL RANGES OF AN OPERATOR

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ABSTRACT. Relative numerical ranges are introduced and their basic properties are listed.

## 1. INTRODUCTION

Let $\mathscr{H}$ be a separable complex Hilbert space. We denote by $\mathscr{S}_{\mathscr{H}}$ the unit sphere of $\mathscr{H}$ and by $\mathcal{B}(\mathscr{H})$ the Banach algebra of all bounded linear operators on $\mathscr{H}$. The numerical range of $S \in \mathcal{B}(\mathscr{H})$ is $W(S) = \{\langle Sx, x\rangle; \quad x \in \mathscr{S}_{\mathscr{H}}\}$. We are interested in some parts of $W(S)$, called relative numerical ranges, which are specified by an operator $T \in \mathcal{B}(\mathscr{H})$. To motivate our definition, we begin with the following lemma.

**Lemma 1.** *Let $\mathscr{K} \neq \{0\}$ be a closed subspace of $\mathscr{H}$ and $P$ be the orthogonal projection onto $\mathscr{K}$. Then, for $S \in \mathcal{B}(\mathscr{H})$, the closure of the numerical range of the compression of $S$ to $\mathcal{K}$ is*

$$(1)\quad \overline{W(PS|_{\mathscr{K}})} = \{\lambda \in \mathbb{C}; \quad \exists\, (x_n)_{n=1}^{\infty} \subseteq \mathscr{S}_{\mathscr{H}} : \lim_{n\to\infty} \|Px_n\| = \|P\| \quad \text{and} \quad \lim_{n\to\infty} \langle Sx_n, x_n\rangle = \lambda\}.$$

The set on the right hand side of (1) has meaning if $P$ is replaced by an arbitrary $T \in \mathcal{B}(\mathscr{H})$. Let

$$(2)\qquad W_T(S) = \{\lambda \in \mathbb{C}; \quad \exists\, (x_n)_{n=1}^{\infty} \subseteq \mathscr{S}_{\mathscr{H}} : \lim_{n\to\infty} \|Tx_n\| = \|T\| \quad \text{and} \quad \lim_{n\to\infty} \langle Sx_n, x_n\rangle = \lambda\}.$$

Following Magajna [2], we call $W_T(S)$ the numerical range of $S$ relative to $T$. In the case $S = T$, (2) reduces to the Stampfli's maximal numerical range of $T$, see [3]. On the other hand, $W_I(S) = \overline{W(S)}$, where $I$ is the identity operator on $\mathscr{H}$. Actually, it is clear from the definition that $W_T(S) = \overline{W(S)}$ for any operator $T$ which is a scalar multiple of an isometry.

Recall that a number $\lambda \in \mathbb{C}$ is an approximate eigenvalue of $T \in \mathcal{B}(\mathscr{H})$ if there exists $(x_n)_{n=1}^{\infty} \subseteq \mathscr{S}_{\mathscr{H}}$, called a sequence of approximate eigenvectors of $T$ at $\lambda$, such that $\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0$. It is obvious that the set $\sigma_{ap}(T)$ of all approximate eigenvalues of $T$ is a part of the spectrum $\sigma(T)$ and it is well-known that $\partial\sigma(T)$, the boundary of $\sigma(T)$, is a subset of $\sigma_{ap}(T)$. In particular, if $T$ is a selfadjoint operator, then $\sigma(T) = \sigma_{ap}(T)$.

Let $|T|$ be the unique positive square root of $T^*T$.

**Lemma 2.** *Let $T \in \mathcal{B}(\mathscr{H})$. If $(x_n)_{n=1}^{\infty} \subseteq \mathscr{S}_{\mathscr{H}}$, then $\lim_{n\to\infty} \|Tx_n\| = \|T\|$ if and only if $(x_n)_{n=1}^{\infty}$ is a sequence of approximate eigenvectors of $|T|$ at $\|T\|$.*

Motivated by Lemma 2 we introduce the following definition.

**Definition 3.** Let $T \in \mathcal{B}(\mathscr{H})$ and $r \in \sigma(|T|)$. The *numerical range of $S \in \mathcal{B}(\mathscr{H})$ relative to $T$ at $r$* is

$$W_T^r(S) = \{\lambda \in \mathbb{C}; \exists\, (x_n)_{n=1}^{\infty} \subseteq \mathscr{S}_{\mathscr{H}} : \lim_{n\to\infty} \||T|x_n - rx_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \langle Sx_n, x_n\rangle = \lambda\}.$$

Note that it follows from the definition that $W_T^r(S) = W_{|T|}^r(S)$. In the following theorem we list properties of the relative numerical ranges.

**Theorem 4.** *Let $S, T \in \mathcal{B}(\mathscr{H})$ and $r \in \sigma(|T|)$.*

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(i) *$W_T^r(S)$ is a non-empty closed convex subset of $\overline{W(S)}$.*
(ii) *$W_T^r(S^*) = \{\overline{\lambda};\ \lambda \in W_T^r(S)\}$ and $W_T^r(\alpha S + \beta I) = \alpha W_T^r(S) + \beta$, where $\alpha, \beta \in \mathbb{C}$ are arbitrary.*
(iii) *Assume that $\gamma \in \mathbb{C}$ is a non-zero number and $V \in \mathcal{B}(\mathscr{H})$ is an isometry. Then $W_{\gamma VT}^{|\gamma| r}(S) = W_T^r(S)$.*
(iv) *If $f$ is a continuous real-valued function on $\sigma(|T|)$, then $W_T^r(S) \subseteq W_{f(|T|)}^{|f(r)|}(S)$ for every $S \in \mathcal{B}(\mathscr{H})$. Moreover, if $f$ is injective and $f(t) \geq 0$ for all $t \in \sigma(|T|)$, then $W_T^r(S) = W_{f(|T|)}^{f(r)}(S)$ for every $S \in \mathcal{B}(\mathscr{H})$.*

## 2. Zero in the relative numerical range

It is known that the position of zero with respect to the numerical range of $S \in \mathcal{B}(\mathscr{H})$ gives some information about $S$. In the following proposition we show that the presence of 0 in $W_T^r(S^*T)$ gives lower bound for the distance from $T$ to the linear space spanned by $S$.

**Proposition 5.** *Let $S, T \in \mathcal{B}(\mathscr{H})$ and $r \in \sigma(|T|)$. If $0 \in W_T^r(S^*T)$, then $\operatorname{dist}(T, \mathbb{C}S) \geq r$. Hence, $\operatorname{dist}(T, \mathbb{C}S) \geq \sup\{r \in \sigma(|T|);\ 0 \in W_T^r(S^*T)\}$.*

For $r = \|T\|$, Proposition 5 implies $\operatorname{dist}(T, \mathbb{C}S) = \|T\|$ whenever $0 \in W_T^{\|T\|}(S^*T)$ because the inequality $\operatorname{dist}(T, \mathbb{C}S) \leq \|T\|$ always holds. Actually the conditions $\operatorname{dist}(T, \mathbb{C}S) = \|T\|$ and $0 \in W_T^{\|T\|}(S^*T)$ are equivalent.

**Theorem 6.** *Let $S, T \in \mathcal{B}(\mathscr{H})$ be arbitrary. Then $\|T\| = \operatorname{dist}(T, \mathbb{C}S)$ if and only if $0 \in W_T^{\|T\|}(S^*T)$.*

We mention a few consequences of Theorem 6.

**Corollary 7.** *Let $V \in \mathcal{B}(\mathscr{H})$ be an isometry and $S \in \mathcal{B}(\mathscr{H})$ be an arbitrary operator. Then $\operatorname{dist}(V, \mathbb{C}S) = 1$ if and only if $0 \in \overline{W(V^*S)}$. In particular, $\operatorname{dist}(I, \mathbb{C}S) = 1$ if and only if $0 \in \overline{W(S)}$.*

**Corollary 8.** *An operator $S \in \mathcal{B}(\mathscr{H})$ is non-invertible if and only if $\operatorname{dist}(U, \mathbb{C}S) = 1$ for every unitary operator $U$.*

**Corollary 9.** *If $S$ is invertible, then there exists a unitary operator $U \in \mathcal{B}(\mathscr{H})$ and a (non-zero) number $\alpha$ such that $\|U^* - \alpha S^{-1}\| < 1$. In particular, if $\|I - S\| < 1$, then there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\|I - \alpha S^{-1}\| < 1$.*

**Corollary 10.** *If $T \in \mathcal{B}(\mathscr{H})$ is invertible, then there exists $\lambda \in \mathbb{C}$ such that $\|T^* - \lambda T^{-1}\| < 1$.*

We close the paper with a result which gives a characterization of $\overline{W(S)} \setminus \sigma(S)$.

**Corollary 11.** *Let $S \in \mathcal{B}(\mathscr{H})$. For $\lambda \in \mathbb{C} \setminus \sigma(S)$, the following assertions are equivalent:*

*(i)* $\lambda \in \overline{W(S)} \setminus \sigma(S)$; *(ii)* $\inf_{\mu \in \mathbb{C}} \|I - \mu(S - \lambda I)^{-1}\| = 1$; *(iii)* $\inf_{\mu \in \mathbb{C}} \|(S - \lambda I)^{-1}(S - \mu I)\| = 1$.

Proofs of all mentioned results can be find in [1].

## References

[1] J. Bračič, C. Diogo, *Relative numerical ranges*, Lin. Alg. Appl., **485** (2015), 208–221.
[2] B. Magajna, *On the distance to finite-dimensional subspaces in operator algebras*, J. London Math. Soc., **47** (1993), 516–532.
[3] J. G. Stampfli, *The norm of a derivation*, Pacific J. Math., **33** (1970), 737–747.

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