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ON AN ACCELERATION TERM IN A DYNAMIC CONTACT OF PLATES WITH A RIGID OBSTACLE

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Abstract. We deal with hyperbolic variational inequalities modeling the behaviour of elastic and viscoelastic anisotropic plates vibrating against a rigid obstacle. We solve the presented initial-boundary value problems by the penalization method. The time derivative of the function representing the nonstationary deflection of the plate middle surface is not continuous due to the hitting the obstacle. The acceleration term appears only implicitly in the viscoelastic case and has the form of a vector measure in the elastic case.

1. Problem formulation and penalization

We deal here with simply supported short memory viscoelastic and elastic anisotropic plate of the constant thickness $h > 0$ vibrating against a rigid obstacle. We have considered similar problems for von Kármán plates in [1] and [2] respectively. We assume a convex bounded region $\Omega \subset \mathbb{R}^2$ to be the middle surface of a plate. Its boundary Γ is C^{Lip}-smooth with an outer unit normal vector $\boldsymbol{n} = (n_1, n_2)$. We set further $I \equiv (0, T)$ a bounded time interval, $Q = I \times \Omega$, $S = I \times \Gamma$ the time-space sets. Due to the rigid obstacle in a form of $x_3 = -h/2$ the initial-boundary value problem for the movement of middle surfaces of plates has the form

$$\left.\begin{aligned} \ddot{u} - a\Delta\ddot{u} + (A_{ijk\ell}\dot{u}_{x_ix_j} + B_{ijk\ell}u_{x_ix_j})_{x_kx_\ell} &= f + g, \\ u \geq 0,\ g \geq 0,\ ug &= 0 \end{aligned}\right\} \text{ on } Q\,, \tag{1}$$

$$u = w, \quad \mathscr{M}(u) := (A_{ijk\ell}\dot{u}_{x_ix_j} + B_{ijk\ell}u_{x_ix_j})n_kn_\ell = 0 \text{ on } S\,, \tag{2}$$

$$u(0,\cdot) = u_0 \geq 0\,,\ \dot{u}(0,\cdot) = v_0 \text{ on } \Omega\,. \tag{3}$$

The Einstein summation convection has been applied above. The unknown function u expresses the deflection of of the middle surface. The plate is acting upon a perpendicular load f and an unknown contact force g between the plate and the contact plane $x_3 = -h/2$. The constant $a > 0$ is a rotary inertia term, $A_{ijk\ell}$, $B_{ijk\ell}$ are the viscoelasticity and elasticity tensors respectively.

For any $\eta > 0$ we define the *penalized problem* in a form

$$\ddot{u} - a\Delta\ddot{u} + (A_{ijk\ell}\dot{u}_{x_ix_j} + B_{ijk\ell}u_{x_ix_j})_{x_kx_\ell} = f + \eta^{-1}u^- \quad \text{on } Q\,, \tag{4}$$

$$u = w, \quad \mathscr{M}(u) := (A_{ijk\ell}\dot{u}_{x_ix_j} + B_{ijk\ell}u_{x_ix_j})n_kn_\ell = 0 \text{ on } S\,, \tag{5}$$

$$u(0,\cdot) = u_0 \geq 0,\ \dot{u}(0,\cdot) = v_0 \text{ on } \Omega\,. \tag{6}$$

We introduce the Hilbert spaces

$$H \equiv L_2(\Omega),\ H^k(\Omega) = \{y \in H:\ D^\alpha y \in H,\ |\alpha| = k\},\ k = 1, 2$$

with the standard inner products $(\cdot,\cdot),\ (\cdot,\cdot)_k,\ k \in \mathbb{N}$, the norms $|\cdot|_0,\ \|\cdot\|_k$ and the space

$$V = H^2(\Omega) \cap \mathring{H}^1(\Omega) = \{y \in H^2(\Omega):\ y = 0 \text{ on } \Gamma\}$$

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with the inner product and the norm

$$((y,z)) = \int_\Omega y_{x_ix_j}(x) z_{x_ix_j}(x)\,dx,\ \ \|y\| = ((y,y))^{1/2},\ y,\, z\in V.$$

We denote by V^* the dual space of linear bounded functionals over V with duality pairing $\langle F, y\rangle_* = F(y),\ F\in V^*,\ y\in V$. It is a Banach space with a norm $\|\cdot\|_*$.
The spaces $V,\ H,\ V^*$ form the Gelfand triple meaning the dense and compact embedings

$$V \hookrightarrow\hookrightarrow H \hookrightarrow\hookrightarrow V^*.$$

For a Hilbert space X we denote by $L_2(I;X)$ the Hilbert space of all functions $y : I\mapsto X$ such that $\|y(\cdot)\|_X \in L_2(0,T)$ with the inner product

$$(u,v)_{L_2(I,X)} = \int_I (u,v)_X\,dt,\ u,\, v\in L_2(I,X),$$

by $L_\infty(I;X)$ the space of essentially bounded functions with values in X, and by $C(\bar I;X)$ the space of continuous functions $y : \bar I\mapsto X,\ \bar I = [0,T]$. For $k\in\mathbb{N}$ we denote by $C^k(\bar I;X)$ the spaces of k-times continuously differentiable functions defined on $\bar I$ with values in X. and we set

$$H^k(I;X) = \{v\in C^{k-1}(\bar I;X) :\ \frac{d^k v}{dt^k}\in L_2(I;X)\}$$

the Hilbert spaces with the inner products

$$(u,v)_{H^k(I,X)} = \int_I [(u,v)_X + \sum_{j=1}^k (u^j, v^j)_X]\,dt,\ k\in\mathbb{N}.$$

If Y is a Banach space, then $L_1(I;X)$ is a Banach space of functions $y : I\mapsto Y$ such that $\|y(\cdot\|_Y \in L_1(0,T)$ with the norm

$$\|y\|_{L_1(I;Y)} = \int_I \|y\|_Y\,dt.$$

2. Solving the viscoelastic problem

We start with the viscoelastic case. It differs from the elastic case in the appearing of nonzero tensor $A_{ijk\ell}$. We assume both tensors to be symmetric and positively definite i.e.

(7) $$A_{ijk\ell} = A_{k\ell ij} = A_{jik\ell},\quad \alpha\,\varepsilon_{ij}\varepsilon_{ij} \le A_{ijk\ell}\,\varepsilon_{ij}\varepsilon_{k\ell}\ \forall\{\varepsilon_{ij}\}\in\mathbb{R}^{2\times 2}_{sym},\ \alpha > 0,$$

(8) $$B_{ijk\ell} = B_{k\ell ij} = B_{jik\ell},\quad \beta\,\varepsilon_{ij}\varepsilon_{ij} \le B_{ijk\ell}\,\varepsilon_{ij}\varepsilon_{k\ell}\ \forall\{\varepsilon_{ij}\}\in\mathbb{R}^{2\times 2}_{sym},\ \beta > 0.$$

We define almost everywhere on $H^2(\Omega)$ the bilinear forms by

(9) $$A(w,y) = A_{ijk\ell} w_{x_ix_j} y_{x_kx_\ell},\ B(w,y) = B_{ijk\ell} w_{x_ix_j} y_{x_kx_\ell}.$$

Further we assume $w\in H^2(\Omega),\ w>0;\ w_{|\Gamma} = u_{0|\Gamma},\ u_0,\ v_0\in V,\ f\in L_2(Q)$.

2.1. **Penalized problem.** We formulate a weak solution of the penalized problem (4)-(6) for the viscoelastic case.

Problem $\mathscr{P}^v_\eta$. *We look for* $u\in L_2(I;V) + w$ *such that* $\dot u\in L_2(I;V),\ \ddot u\in L_2(I;\mathring{H}^1(\Omega))$, *the equation*

(10) $$\int_Q \left[\ddot u z + \nabla\ddot u\cdot\nabla z + A(\dot u, z) + B(u,z) - \eta^{-1}u^- z\right]\,dx\,dt = \int_Q fz\,dx\,dt$$

holds for all $z\in L_2(I;V)$ *together with the initial conditions* (6).

We derive the *a priori estimates* for solutions of the Problem $\mathscr{P}^v_\eta$. We insert

$$z = \begin{cases}\dot u \text{ for } t\le s\\ 0 \text{ for } t > s\end{cases}$$

in (10) for arbitrary $s \in I$, denote $Q_s = (0,s)\times\Omega$ and obtain

$$\int_{Q_s}\Big[\frac{1}{2}\partial_t\left(\dot u^2 + a|\nabla\dot u|^2 + B(u,u) + \eta^{-1}(u^-)^2\right) + A(\dot u,\dot u)\Big]\,dx\,dt = \int_{Q_s} f\dot u\,dx\,dt. \tag{11}$$

Applying the coercivity assumptions (7), (8) we obtain the $\eta-$independent *a priori* estimates

$$\|\dot u\|_{L_2(I,V)} + \|\dot u\|_{L_\infty(I,H^1(\Omega))} + \|u\|_{L_\infty(I,V)} \le C_1 \equiv C_1(f,u_0,v_0,w) \tag{12}$$

and formulate the existence and uniqueness theorem of a solution to the penalized problem.

Theorem 1. *There exists a unique solution of the problem* $\mathscr{P}^v_\eta$.

Proof. Let $\{w_i \in V;\ i\in\mathbb{N}\}$ be an orthonormal basis of V. We construct the Galerkin approximation u_m of a solution in a form

$$u_m(t) = \sum_{i=1}^m \alpha_i(t)w_i,\ \ \alpha_i(t)\in\mathbb{R},\ \ i=1,...,m,\ \ m\in\mathbb{N}$$

given by the solution of the approximated problem

$$\begin{aligned}&\int_\Omega \big(a\nabla\ddot u_m(t)\cdot\nabla w_i + \ddot u_m(t)w_i + A(\dot u_m(t),w_i) + B(u_m(t),w_i) - \eta^{-1}u_m(t)^- w_i\big)\,dx\\ &= \int_\Omega f(t)w_i\,dx,\ \ i=1,...,m,\end{aligned} \tag{13}$$

$$u_m(0) = u_{0m},\ \ \dot u_m(0) = v_{0m};\ \ u_{0m}\to u_0,\ \ u_{0m}\to v_0 \text{ in } V. \tag{14}$$

Applying the estimates (12) we obtain in the same way as in [1] the subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$), and a function u such that the following convergences

$$\begin{aligned}
u_m &\rightharpoonup^* u && \text{in } L_\infty(I;V),\\
\dot u_m &\rightharpoonup^* \dot u && \text{in } L_\infty(I;H^1(\Omega)),\\
\dot u_m &\rightharpoonup \dot u && \text{in } L_2(I;V),\\
\ddot u_m &\rightharpoonup \ddot u && \text{in } L_2(I;H^1(\Omega)),\\
\dot u_m &\to \dot u && \text{in } L_2(I;H^{2-\varepsilon}(\Omega))\ \forall\varepsilon\in(0,1)
\end{aligned} \tag{15}$$

hold and u solves (10). The initial conditions (6) follow due to (14) and the existence part of the proof is completed. The uniqueness follows using the Gronwall lemma.

2.2. **The original problem.** To state the variational formulation of this problem we shall use the shifted cone

$$\mathscr{K} := \{y\in L_2(I;V)+w;\ \dot y\in L_2(I;H^1(\Omega)),\ y\ge 0\}. \tag{16}$$

Performing the integration by parts both with respect to t and x we obtain the formulation without an acceleration term.

Problem $\mathscr{P}^v$ *We look for* $u\in\mathscr{K}$ *such that* $\dot u\in L_2(I;V)$ *and the inequality*

$$\begin{aligned}&\int_Q \big(A(\dot u, y-u) + B(u,y-u) - a\nabla\dot u\cdot\nabla(\dot y-\dot u) - \dot u(\dot y-\dot u)\big)\,dx\,dt\\ &\quad+\int_\Omega \big(a\nabla\dot u\cdot\nabla(y-u) + \dot u(y-u)\big)(T,\cdot)\,dx\\ &\ge \int_\Omega\big(a\nabla v_0\cdot(\nabla y(0,\cdot)-\nabla u_0) + v_0(y(0,\cdot)-u_0)\big)\,dx + \int_Q f(y-u)\,dx\,dt.\end{aligned} \tag{17}$$

holds for any $y\in\mathscr{K}$.

Using the solutions of the penalized problem $\mathscr{P}^v_\eta$, $\eta>0$, we verify the following existence theorem.

Theorem 2. *There exists a solution of the Problem* $\mathscr{P}^v$.

Proof. The solution $\{u_\eta\}$ of $\mathscr{P}^v_\eta$ fulfils the η-independent estimate

$$\|\dot u_\eta\|_{L_2(I,V)} + \|\dot u_\eta\|_{L_\infty(I,H^1(\Omega))} + \|u_\eta\|_{L_\infty(I,V)} \le C_1 \equiv C(f,u_0,v_0,w) \tag{18}$$

After putting $u = u_\eta,\ z = w - u_\eta$ in (10) we obtain in the same way as in the case of clamped plate in [1] the following crucial estimate

$$\|\eta^{-1}u^-_\eta\|_{L_1(Q)} + \| - a\Delta\ddot u_\eta + \ddot u_\eta\|_{L_1(I;V^*)} \le C_2(f,u_0,v_0,w) \tag{19}$$

After applying the generalization of the Aubin lemma derived in [6], the relative compactness of the sequence $\{-a\Delta\ddot u_{\eta_k} + \ddot u_{\eta_k}\},\ \eta_k \to 0+$ implies the strong convergence $\dot u_k \to \dot u$ in $L_2(I;H^1(\Omega)$. The other convergences are of the same type as in the first triple of (15) and the existence of a solution follows from the penalized problems $\mathscr{P}^v_\eta,\ \eta > 0$.

3. Solving the elastic problem

We assume $B_{ijk\ell} = 0,\ i,j,k,\ell \in \{1,2\}$ in this case.

3.1. **Elastic penalized problem.** We formulate a weak solution of the penalized problem (4)-(6) for the elastic case.

Problem $\mathscr{P}^e_\eta$. We look for $u \in L_2(I;V) + w$ such that $\ddot u \in L_2(Q)$, the equation

$$\int_Q \left[\ddot u(z - \Delta z) + B(u,z) - \eta^{-1}u^- z\right]\,dx\,dt = \int_Q fz\,dx\,dt \tag{20}$$

holds for all $z \in L_2(I;V)$ together with the initial conditions (6).

In the same way as in the viscoelastic case we obtain η-independent *a priori* estimates of solutions $u \equiv u_\eta$:

$$\|\dot u\|_{L_\infty(I,H^1(\Omega))} + \|u\|_{L_\infty(I,V)} \le C_3 \equiv C_3(f,u_0,v_0,w) \tag{21}$$

and formulate the existence and uniqueness theorem of a solution to the penalized problem.

Theorem 3. *There exists a unique solution of the problem $\mathscr{P}^e_\eta$.*

3.2. **The elastic contact problem.** In this case we do not have the strong convergence of a sequence of time derivatives $\dot u_{\eta_k}$ as in the viscoelastic case. The $L_1(Q)$ estimate of the penalty term implies the boundedness of the corresponding acceleration terms in the space $\mathscr{M}(I;L_2(\Omega))$ of vector measures (see [3] for details).

We introduce the shifted cone

$$\mathscr{C} := \{y \in C_w(\bar I;V) + w;\ y \ge 0\}, \tag{22}$$

where $C_w(\bar I;V)$ is the set of weakly continuous functions mapping the time interval $\bar I$ into V. We are looking for a solution in the shifted cone $\mathscr{Y} = \{u \in w + \mathscr{W}\}$ with the set

$$\mathscr{W} = \{v \in C_w(\bar I;V),\ \dot v \in R_w(\bar I,\mathring H^1(\Omega)),\ \ddot v \in \mathscr{M}_0(Q)\}. \tag{23}$$

We denote by $R_w(\bar I,\mathring H^1(\Omega))$ the set of all weakly right continuous weakly regulated maps mapping $\bar I$ to $\mathring H^1(\Omega))$ and by $\mathscr{M}_0(Q)$ the set of signed measures M on Q fulfilling

$$\left|\int_Q \varphi\,dM\right| \le c|\varphi|_{C_0(I;L_2(\Omega))}\ \forall\varphi \in C_0(Q).$$

We remark that $C_0(I;L_2(\Omega))$ and $C_0(Q)$ are the sets of continuous functions from $\mathbb{R}$ to $L_2(\Omega)$ vanishing outside I and from $\mathbb{R}^3$ to $\mathbb{R}$ vanishing outside Q respectively.

Problem $\mathscr{P}^e$ *To find $u \in \mathscr{Y}$ such that the inequality*

$$\int_Q (\mathbf{1} - a\Delta)(y-u)\,d\ddot u \ge \int_Q [f(y-u) - B(u,y-u)]\,dx\,dt \tag{24}$$

holds for all $y \in \mathscr{C}$ and the initial conditions (3) *are fulfilled.*

Using the penalized Problem $\mathscr{P}^e_\eta$ with the estimates (21) and the $L_1(Q)$ estimate of the penalty term we obtain after applying the technique of vector measures from [3] (see also [4], [7]) the existence of a solution in

Theorem 4. *There exists a solution of the Problem $\mathscr{P}^e$.*

Remark 5. The set $\mathscr{W}$ in (23) can be experssd also in the form

$$\mathscr{W} = \{v \in L_\infty(I;V):\ \dot{v} \in L_\infty(I, \mathring{H}^1(\Omega)),\ \ddot{v} \in \mathscr{M}_0(Q)\}.$$

The representation (23) enables to express directly the initial conditions (3).

Remark 6. The interpretation of accelerations through vector measures is possible also in the viscoelastic case.

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