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ON THE LANDESMAN-LAZER CONDITION FOR THREE POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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The main goal of this article is to derive sufficient conditions of Landesman-Lazer type for the existence of a solution of a special type of generalized boundary value problem for a second order differential equation with possibly unbounded nonlinearity. Generalized boundary conditions are given by continuous linear functionals. In such case the notion of adjoint problem to the associated linear problem cannot be expressed by a classical differential equation[5]. We use duality principle instead of adjointness. The kernel of linear operator defined by the problem below is one dimensional. To derive a condition of Landesman-Lazer type [2], we use Ljapunov -Schmidt decomposition and Leray-Schauder degree [1], [6].

We consider the nonlinear generalized boundary value problem

$$\begin{aligned} x'' + x + f(t,x) &= h(t), \\ x(0) = x(2\pi), \qquad x'(t_0) &= 0, \end{aligned} \tag{1}$$

where $I = [0, 2\pi]$, $f : I \times R \to R$ is a continuous function and $h : L^1(I)$.

A solution $x(t)$ is a function, $x \in C^1(I)$, $x'' \in L^1(I)$.

We denote $Z = L^1(I)$, $Y = \{y \in C^1(I);\ y'' \in L^1(I), y(0) = y(2\pi),\ y'(t_0) = 0\}$.

We use the bilinear duality functional $D : (Z, Y) \to R$ given by

$$D(z, y) = \int_0^{t_0} z(t)y(t_0 - t)\,dt + \int_{t_0}^{2\pi} z(t)y(2\pi + t_0 - t)\,dt\,.$$

By the linear part of boundary value problem (1)

$$\begin{aligned} x'' + x &= h(t)\,, \\ x(0) = x(2\pi), \qquad x'(t_0) &= 0\,, \end{aligned} \tag{2}$$

is defined the linear operator $L : Y \to Z$

$$Lx = x'' + x\,.$$

A simple computation leads to $D(Lx, y) = D(Ly, x)$ for each $x, y \in Y$. This identity together with the separation property of D means that the linear three point boundary value problem is selfdual.

The kernel of L is one dimensional, $N(L) = [\cos(t - t_0)]$. As the operator L is Fredholm of index zero [3], the codimension of ImL is one and Z=Im$L \oplus Z_2$ with $Z_2 = [\sin(t - 2t_0)]$.

Let $X = C(I)$ and $N : X \to Z$ be the nonlinear operator defined by $N(x) = f(t, x(t))$.

Then the problem (1) can be written in the form

$$Lx + N(x) = h\,. \tag{3}$$

The spaces X, Y and Z are decomposed to direct sums $X = N(L) \oplus X_2$, $Y = N(L) \oplus Y_2$, $Z = Z_2 \oplus \mathrm{Im}L$, $\dim N(L) = \dim Z_2 = 1$. We define the natural isomorphism $J : Z_2 \to N(L)$ by $J(k\cos(t - t_0)) = k\sin(t - 2t_0)$ and a projection $Q : Z \to Z_2$ given by $Q(z) = \frac{1}{\pi}\int_0^{2\pi} z(t)\sin(t - 2t_0)\,dt\sin(t - 2t_0)$.

The following existence theorem holds.

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Theorem 1. *Suppose that*

(i) $f : I \times R \to R$ *is a continuous function of sublinear growth, i.e. the following a priori estimation holds*

$$\exists C > 0,\ 0 < \alpha < 1\ , r > 0 \text{ such that } \forall t \in I,\ |x| > r;\ |f(t,x)| \le C|x|^\alpha\,,$$

(ii) *there exist* $f_+, f_- \in L^1(I)$ *such that*

$$f_+(t) = \lim_{x\to\infty} f(t,x)\,, \qquad f_-(t) = \lim_{x\to-\infty} f(t,x)\,,$$

(iii) *denoting* $I_{t_0+} = \{t \in I;\ \cos(t-t_0) \ge 0\}$, $I_{t_0-} = \{t \in I;\ \cos(t-t_0) \le 0\}$ *the following Landesman-Lazer type condition holds*

$$\int_{I_{t_0+}} f_+(t)\sin(t-2t_0)\,dt + \int_{I_{t_0-}} f_-(t)\sin(t-2t_0)\,dt > \int_0^{2\pi} h(t)\sin(t-2t_0)\,dt >$$
$$> \int_{I_{t_0+}} f_-(t)\sin(t-2t_0)\,dt + \int_{I_{t_0-}} f_+(t)\sin(t-2t_0)\,dt\,.$$

Then the three point boundary value problem (1) is solvable for each $h \in L^(I)$ *satisfying (iii).*

Proof. The operator equation (3) is equivalent to the fixed point problem

$$x = T(x)\,, \tag{4}$$

where $T : X \to X$ is defined as

$$T(x) = Px - JQ(N(x)-h) - L_p^{-1}(I-Q)(N(x)-h)\,. \tag{5}$$

Here $P : X \to N(L)$ is a projection $Q : Z \to Z_2$ given by $P(x) = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos(t-t_0)\,dt\cos(t-t_0)$ and $L_p^{-1} : \mathrm{Im}L \to Y_2$ is the inverse operator to the restriction $L|_{Y_2}$ which is continuous as the operator $\mathrm{Im}L \to Y_2$ and compact as $\mathrm{Im}L \to X$. We embed the equation (4) to the homotopic system of equations

$$H(\lambda, x) = 0\,, \tag{6}$$

where $H : [0,1] \times Z \to Z$ is defined as $H(\lambda, x) = I - \lambda T(x)$.

The pair (λ, x) is a solution of (6) iff $x = x_1 + x_2 \in N(L) \oplus Y_2$ and

$$\begin{gathered} x_2 + \lambda L_p^{-1}(I-Q)(N(x)-h) = 0\,, \\ (1-\lambda)x_1 + \lambda JQ(N(x)-h) = 0\,. \end{gathered} \tag{7}$$

A solution of (7) is for $\lambda = 1$ also solution of (4).

We prove that each possible solution of (7) is bounded. Suppose the opposite that there is a sequence of solutions x_n of (7) $x_n = x_{1n} + x_{2n}$, $x_{1n} \in D(L)$, $x_{2n} \in Y_2$ such that $\|x_n\| \to \infty$. By $\|\cdot\|$ we denote the norm in $C(I)$. The growth assumption on f leads to $\|x_{2n}\| \le c\|x_n\|^\alpha$, and moreover $\|x_{2n}\| \le c\|x_{1n}^\alpha\|$ for suitable $c > 0$ and n sufficiently large.

As $x_n = c_n\cos(t-t_0) + x_{2n}$, then either $c_n \to \infty$ and $f(t, x_n(t)) \to f_+(t)$ for $t \in I_{t_0+}$, $f(t, x_n(t)) \to f_-(t)$ for $t \in I_{t_0-}$

or $c_n \to -\infty$ and $f(t, x_n(t)) \to f_-(t)$ for $t \in I_{t_0+}$, $f(t, x_n(t)) \to f_+(t)$ for $t \in I_{t_0-}$.

Both possibilities are in a contradiction with Landesman-Lazer condition (iii).

That means $\|x\|$ is bounded for each solution x of (7) by a constant $R > 0$ independently on λ.

Set $\Omega = \{x \in X;\ ||x|| < R\}$. Now either (6) possesses a solution for $\lambda = 1$ on the set $\partial\Omega$ or the Leray-Schauder degree of H is well defined on Ω for each $0 \le \lambda \le 1$ and

$$d(I - \lambda T, \Omega, 0) = d(I, \Omega, 0) = 1\,.$$

In both cases (4) is solvable on $\bar\Omega$ and its solution $x(t)$ is a solution of the three point boundary value problem (1). □

The existence of a solution of (1) can be proved also in the case when limit functions f_+, f_- are not integrable.

Theorem 2. *Suppose that condition* (i) *from Theorem* 1 *holds and*

(ii)
$$\lim_{x\to\infty} f(t,x) = f_+(t) \text{ for each } t\in I\setminus I_1,\ \text{ where } f_+\in L^1(I\setminus I_1)$$
$$\lim_{x\to-\infty} f(t,x) = f_-(t) \text{ for each } t\in I\setminus I_2,\ \text{ where } f_-\in L^1(I\setminus I_2),$$
where $I_1\subset\{t\in I;\ \cos(t-t_0)>0\ \wedge\sin(t-2t_0)>0\}$ $I_2=\{t\in I;\ \cos(t-t_0)<0\ \wedge\sin(t-2t_0)<0\}$ *with* $\mu(I_1\cup I_2)>0$.

(iii)
$$\lim_{x\to\infty} f(t,x)=\infty\ \forall t\in I_1\,,\qquad \lim_{x\to-\infty} f(t,x)=-\infty\ \forall t\in I_2\,,$$

(iv)
$$\int_0^{2\pi} h(t)\sin(t-2t_0)\,dt > \int_{I_{t_0+}} f_-(t)\sin(t-2t_0)\,dt + \int_{I_{t_0-}} f_+(t)\sin(t-2t_0)\,dt.$$

Then three point boundary value problem (1) *is solvable for each* $h\in L^1(I)$ *satisfying* (iv).

Theorem 3. *Suppose that condition* (i) *from Theorem* 1 *holds and*

(ii)
$$\lim_{x\to\infty} f(t,x) = f_+(t) \text{ for each } t\in I\setminus I_3\,,\ \text{ where } f_+\in L^1(I\setminus I_3)$$
$$\lim_{x\to-\infty} f(t,x) = f_-(t) \text{ for each } t\in I\setminus I_4\,,\ \text{ where } f_-\in L^1(I\setminus I_4)\,,$$
where $I_3\subset\{t\in I;\ \cos(t-t_0)>0\ \wedge\sin(t-2t_0)<0\}$, $I_4=\{t\in I;\ \cos(t-t_0)<0\ \wedge\sin(t-2t_0)>0\}$ *with* $\mu(I_3\cup I_4)>0$.

(iii)
$$\lim_{x\to\infty} f(t,x)=-\infty\ \forall t\in I_3\,,$$
$$\lim_{x\to-\infty} f(t,x)=\infty\ \forall t\in I_4\,,$$

(iv)
$$\int_{I_{t_0+}} f_+(t)\sin(t-2t_0)\,dt + \int_{I_{t_0-}} f_-(t)\sin(t-2t_0)\,dt > \int_0^{2\pi} h(t)\sin(t-2t_0)\,dt\,.$$

Then three point boundary value problem (1) *is solvable for each* $h\in L^1(I)$ *satisfying* (iv).

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