10th Workshop Functional Analysis and its Applications, September 7–12, 2015, Kočovce, 31–33

A NOTE ON THE REACHABILITY OF A FIBONACCI CONTROL SYSTEM

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Abstract. Motivated by applications in robotics, we investigate a discrete control system related Fibonacci sequence and we characterize its reachable set.

This note is devoted to the characterization of the reachable set of the discrete control system

$$\text{(F)} \qquad \begin{cases} x_0 = u_0 \\ x_1 = u_1 + \frac{u_0}{q} \\ x_{n+2} = u_{n+2} + \frac{x_{n+1}}{q} + \frac{x_n}{q^2}. \end{cases}$$

Motivations in investigating the systems of the form (F) come from robotics, indeed it can be shown that x_n represents the total length of a telescopic, self-similar robotic arm [LLV, LLVa].

By an inductive argument we get the closed formula

$$x_n = x_n(u) = \sum_{k=0}^{n} \frac{f_k}{q^k} u_{n-k},$$

where $f_k = f_{k-1} + f_{k-2}, f_1 = f_0 = 1$ denotes Fibonacci sequence [LLVa]. Consequently the *asymptotic reachable set* $\mathcal{R}_q$ of the system (F) reads

$$\mathcal{R}_q := \{\lim_{n\to\infty} x_n(u) \mid u \in \{0,1\}^\infty\} = \left\{ \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k \mid u_k \in \{0,1\} \right\}. \tag{1}$$

Remark 1. The set $\mathcal{R}_q$ is well defined if and only if the scaling ratio q is greater than the Golden Mean φ, this indeed ensures the convergence of the series $\sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k$.

In order to give a full description of $\mathcal{R}_q$, we shall make use of the following definitions

$$\begin{aligned} \mathcal{R}_{q,j} &:= \left\{ \sum_{k=0}^{\infty} \frac{f_{j+k}}{q^k} u_k \mid u_k \in \{0,1\} \right\} \\ S(q,j) &:= \sum_{k=0}^{\infty} \frac{f_{j+k}}{q^k} = \frac{f_j q^2 + f_{j-1} q}{q^2 - q - 1} \\ Q(j) &:= \text{greatest solution of the equation } S(q, j+1) = q f_j \\ &= \frac{1}{2 f_j} (f_{j+2} + \sqrt{f_{j+2}^2 + 8 f_j^2}). \end{aligned}$$

Notice that, as $j \to \infty$, for all $q > \varphi$, $S(q,j) \uparrow \infty$ while $Q(j) \uparrow \frac{1}{2}(\varphi^2 + \sqrt{\varphi^2 + 1})$. Also notice the recursive relation

$$S(q,j) = q(S(q, j-1) - f_{j-1}). \tag{2}$$

Lemma 2. *If $q \in (\varphi, Q(j)]$ then $\mathcal{R}_{q,j} = [0, S(q,j)]$.*

2010 *Mathematics Subject Classification.* 70E60,11A63.

Proof. We show the claim by double inclusion. The inclusion $\mathcal{R}_{q,j} \subseteq [0, S(q,j)]$ readily follows by the definitions of $\mathcal{R}_{q,j}$ and of $S(q,j)$. To show the other inclusion, for all $x \in [0, S(q,j)]$ we consider the sequences (r_h) and (u_h) defined by

$$\begin{cases} r_0 = x; \\ u_h = \begin{cases} 1 & \text{if } r_h \in [f_{j+h}, S(q, j+h)] \\ 0 & \text{otherwise} \end{cases} \\ r_{h+1} = q(r_h - u_h f_{j+h}) \end{cases} \tag{3}$$

We show by induction

$$x = \sum_{k=0}^{h} \frac{f_{j+k}}{q^k} u_k + \frac{r_{h+1}}{q^{h+1}} \quad \text{for all } h \geq 0. \tag{4}$$

For $h = 0$ one has $r_1 = q(x - u_0 f_k)$ and consequently $x = f_k u_0 + r_1/q$. Assume now (4) as inductive hypothesis. Then

$$r_{h+2} = q^{h+2}\left(x - \sum_{k=0}^{h} \frac{f_{j+k}}{q^k} u_k\right) - q f_{j+h+1} u_{h+1}$$

Consequently,

$$x = \sum_{k=0}^{h+1} \frac{f_{j+k}}{q^k} u_k + \frac{r_{h+2}}{q^{h+2}}$$

and this completes the proof of the inductive step and, therefore, of (4).

Now we claim that if $q \leq Q(j)$ then

$$r_h \in [0, S(q, j+h)] \quad \text{for every } h. \tag{5}$$

We show the above inclusion by induction. If $h = 0$ then the claim follows by the definition of r_0 and by the fact that $x \in [0, S(q,j)]$. Assume now (5) as inductive hypothesis and notice that the definition of $S(q, j+h)$ implies $f_{j+h} \leq S(q, j+h)$. If $r_h \in [0, f_{j+h})$ then $r_{h+1} = qr_h \in [0, qf_{j+h}] \subseteq [0, S(q, j+h+1)]$ – where the last inclusion follows by the definition of $Q(j+h)$ and by the fact that $q \leq Q(j) < Q(j+h)$. If otherwise $r_h \in [f_{j+h}, S(q, j+h)]$ then $r_{h+1} = q(r_h - f_{j+h}) \subseteq [0, q(S(q, j+h) - f_{j+h})] = [0, S(q, j+h+1)]$ (see (2)) and this completes the proof of (5).

Recalling $f_n \sim \varphi^n$ as $n \to \infty$, one has

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f_{j+k}}{q^k} u_k &= \lim_{h\to\infty} \sum_{k=0}^{h-1} \frac{f_{j+k}}{q^k} u_k \overset{(4)}{=} x - \lim_{h\to\infty} \frac{r_h}{q^h} \overset{(5)}{\geq} x - \lim_{h\to\infty} \frac{S(q, j+h)}{q^h} \\ &= x - \lim_{h\to\infty} \frac{q^2 f_{h+j+1} + q f_{j+h}}{q^{j+h}(q^2 - q - 1)} = x. \end{aligned}$$

On the other hand

$$\sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k = x - \lim_{h\to\infty} \frac{r_h}{q^h} \leq x$$

and this proves $x = \sum_{h=0}^{\infty} \frac{f_{h+k}}{q^h} u_h$. It follows by the arbitrariety of x that $[0, S(q,k)] \subseteq \mathcal{R}_{q,k}$ and this concludes the proof. □

Remark 3. By applying Lemma 2 to the case $j = 0$ we get that if $q \in (\varphi, Q(0)]$ then $\mathcal{R}_q = [0, S(q, 0)]$. This result was already proved in [LLVa].

Theorem 4. *For all $j \geq 1$ if $q \in (Q(j-1), Q(j)]$ then $\mathcal{R}_q$ is composed by the disjoint union of 2^j intervals, in particular*

$$\mathcal{R}_q = \bigcup_{u_0,\dots,u_{j-1}\in\{0,1\}} \left[\sum_{k=0}^{j-1} \frac{f_k}{q^k} u_k, \sum_{k=0}^{j-1} \frac{f_k}{q^k} u_k + S(q,j)\right]. \tag{6}$$

Moreover if $q \geq \frac{1}{2}(\varphi^2 + \sqrt{\varphi^2+1})$ then the map $u \mapsto x_u = \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k$ is increasing with respect to the lexicographic order and $\mathcal{R}_q$ is a totally disconnected set.

Proof. Fix $j \geq 1$ and let $q \in (Q(j-1), Q(j)]$. First of all we notice that

$$\mathcal{R}_q = \left\{ \sum_{k=0}^{\infty} \frac{f_k}{q^k} u_k \mid u_k \in \{0,1\} \right\} = \bigcup_{u_0,\dots,u_{j-1}\in\{0,1\}} \sum_{k=0}^{j-1} \frac{f_k}{q^k} u_k + \frac{1}{q^j} \mathcal{R}_{q,j}.$$

Since $q \leq Q(j)$ then by Lemma 2 we have $R_{q,j} = [0, S(q,j)]$ and this implies (6). We now want to prove that the union in (6) is disjoint. To this end consider two binary sequences $(v_0, \dots, v_{j-1})$ and $(u_0, \dots, u_{j-1})$ and assume $(v_0, \dots, v_{j-1}) > (u_0, \dots, u_{j-1})$ in the lexicographic order. Let $h \in \{0, \dots, j-1\}$ be the smallest integer such that $v_h = 1$ and $u_h = 0$. Then $q > Q(j-1) \geq Q(h)$ implies

$$\sum_{k=0}^{j-1} \frac{f_k}{q^k} v_k - \left(\sum_{k=0}^{j-1} \frac{f_k}{q^k} u_k + \frac{S(q,j)}{q^j} \right) \geq \frac{f_h}{q^h} - \sum_{k=h+1}^{j-1} \frac{f_k}{q^k} + \frac{S(q,j)}{q^j} = \frac{f_h}{q^h} - \frac{S(q,h+1)}{q^{h+1}} > 0$$

and, consequently, that the union in (6) is disjoint. To show the second part of the claim we assume $q \geq \frac{1}{2}(\varphi^2 + \sqrt{\varphi^2+1})$ and we let $u = (u_0, \dots, u_n, \dots,)$ and $v = (v_0, \dots, u_n, \dots,)$ be two infinite binary sequences such that $v > u$ in the lexicographic order. As above let h be the smallest integer such that $0 = u_h < v_h = 1$ and define $x_\nu = \sum_{k=0}^{\infty} \frac{f_k}{q^k} \nu_k$ with $\nu \in \{u, v\}$. One has

$$x_v - x_u = \frac{f_h}{q^h} + \sum_{k=h+1}^{\infty} \frac{f_k}{q^k} v_k - \sum_{k=h+1}^{\infty} \frac{f_k}{q^k} u_k \geq \frac{f_h}{q^h} - \frac{1}{q^{h+1}} S(q, h+1) > 0 \tag{7}$$

Indeed $q \geq \frac{1}{2}(\varphi^2 + \sqrt{\varphi^2+1})$ implies $q > Q(h)$ for all $h \geq 0$. This implies that the map $\nu \mapsto x_\nu$ is increasing with respect to the lexicographic order. As a consequence, for all $x_w \in \mathcal{R}_q$ such that $x_u < x_w < x_v$ one has $u < w < v$ in the lexicographic order. In particular $w_j = u_j = v_j$ for $j = 0, \dots, h-1$, $w_h = u_h = 0$ and $(w_{h+1}, \dots, w_{h+n}, \dots) > (u_{h+1}, \dots, u_{h+n}, \dots)$. Therefore $x_w = \sum_{k=0}^{h-1} \frac{f_k}{q^k} u_k + \delta_w$ and $\delta_w = \sum_{k=h+1}^{\infty} \frac{f_k}{q^k} w_k \leq \frac{1}{q^{h+1}} S(q, h+1)$. On the other hand the last inequality in (7) implies that we may choose some

$$\delta \in \left(\frac{1}{q^{h+1}} S(q, h+1), \frac{f_h}{q^h} \right)$$

and setting $x := x_u + \delta$ we get $x_u < x < x_v$ and, in view of above reasoning, $x \notin \mathcal{R}_q$. By the arbitrariety of x_u and x_v we deduce that $\mathcal{R}_q$ is a totally disconnected set and this completes the proof. □

References

[LLV] A. C. Lai, P. Loreti and P. Vellucci. A model for Robotic Hand based on Fibonacci sequence. *Proceedings of the 11-th International Conference on Informatics in Control, Automation and Robotics, 577–587*, 2014.

[LLVa] A. C. Lai, P. Loreti and P. Vellucci. A Fibonacci control system. *arXiv preprint*, arXiv:1403.2882, 2014.

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