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IRREDUCIBLE AND TOTALLY IRREDUCIBLE REPRESENTATIONS OF ALGEBRAS. PROBLEM OF FELL AND DORAN.

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Abstract. The long-standing problem of Fell and Doran concerns total irreducibility of an irreducible representation of an algebra on a topological vector space. Żelazko proved a theorem which gives a necessary and sufficient condition for the positive solution of this problem. We discuss some consequences of Żelazko's theorem.

Let X be a vector space over the field $\mathbb{K}$ of real or complex numbers. Denote by $\mathcal{L}(X)$ the algebra of all endomorphisms of X. A representation of an associative algebra $\mathcal{A}$ over $\mathbb{K}$ on the space X is a homomorphism $T\colon \mathcal{A} \to \mathcal{L}(X)$, $a \mapsto T_a$. A representation (T, X) of $\mathcal{A}$ is algebraically irreducible if for every non-zero $x \in X$ the orbit $\mathcal{O}(T; x) = \{T_a x\colon\ a \in \mathcal{A}\}$ coincides with X. By $(T^{(k)}, X^k)$ we denote the representation of $\mathcal{A}$ in the k-fold direct sum of X defined by

$$T_a^{(k)}(x_1, \dots, x_k) = (T_a x_1, \dots, T_a x_k).$$

A representation (T, X) is called totally algebraically irreducible if for every $k \in \mathbb{N}$ and for every k-tuple $\mathbf{x} = (x_1, \dots, x_k)$ of linearly independent elements the orbit $\mathcal{O}(T^{(k)}; \mathbf{x})$ is equal to X^k. We denote by $\mathcal{T}$ the set $\{T_a\colon\ a \in \mathcal{A}\}$ and by $\mathcal{T}'_{\mathrm{alg}}$ its (algebraic) commutant, i.e.

$$\mathcal{T}'_{\mathrm{alg}} = \{A \in \mathcal{L}(X)\colon\ AT_a = T_a A, a \in \mathcal{A}\}.$$

The classical Jacobson Density Theorem [2] says the following:

Theorem 1. *If (T, X) is an algebraically irreducible representation of an algebra $\mathcal{A}$ on the vector space X and $\mathcal{T}'_{\mathrm{alg}} = \mathbb{K}\,\mathrm{id}_X$, then (T, X) is totally algebraically irreducible.*

Now we pass to the case when X is a topological vector space. The algebra of all linear continuous operators on X is denoted by $L(X)$. By a representation of an algebra $\mathcal{A}$ on X we mean a homomorphism of $\mathcal{A}$ into $L(X)$. A representation (T, X) of the algebra $\mathcal{A}$ on X is called irreducible if for every $x \neq 0$ the orbit $\mathcal{O}(T; x)$ is dense in X. The commutant $\mathcal{T}'$ in this case is defined to be the set $\{A \in L(X)\colon\ AT_a = T_a A, a \in \mathcal{A}\}$. The representation (T, X) is totally irreducible if for every $k \in \mathbb{N}$ and every linearly independent system $\mathbf{x} = (x_1, \dots, x_k) \in X^k$ the orbit $\mathcal{O}(T^{(k)}; \mathbf{x})$ is dense in X^k, in other words the set $\mathcal{T}$ is dense in $L(X)$ in the strong operator topology.

The counterpart of the Jacobson Density Theorem in case $\mathcal{A}$ is a Banach algebra and X is a Banach space is the following:

Theorem 2. *Let $\mathcal{A}$ be a Banach algebra over $\mathbb{C}$ and let X be a complex Banach space. Let (T, X) be an algebraically irreducible representation of $\mathcal{A}$ on X, then (T, X) is totally algebraically irreducible.*

In the monograph [1] Fell and Doran formulated a problem which can be considered to be the topological version of the Jacobson Density Theorem.

Problem (Fell and Doran)**.** Let (T, X) be a representation of an algebra $\mathcal{A}$ on a topological vector space X such that

(i) (T, X) is irreducible;
(ii) $\mathcal{T}' = \mathbb{K}\,\mathrm{id}_X$.

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Is the representation (T, X) totally irreducible?

Notice that condition (ii) is necessary, for if T is an operator on X without non-trivial invariant subspace, then the algebra $\mathcal{P}(T)$ of polynomial functions of T acts on X irreducibly. On the other hand this algebra is commutative, hence it cannot be dense in $L(X)$.

The most desirable form of a solution of the problem of Fell and Doran would be description of the class of topological vector spaces X, say $(\mathcal{FD})$-class, for which the above assumptions (i) and (ii) on the representation (T, X) assure its total irreducibility. Certainly, the class $(\mathcal{FD})$ contains all finite dimensional spaces. Żelazko in [5] proved that the space (s) of all complex sequences provided with the topology of pointwise convergence belongs to the class $(\mathcal{FD})$. Surprisingly it is the only one infinite dimensional space known to be a member of the class $(\mathcal{FD})$.

A specially important contribution to the study of the problem of Fell and Doran is another paper of Żelazko [4]. To explain the main result of this paper we need the following definition.

If (T, X) and (S, Y) are representations of an algebra $\mathcal{A}$ and R is a densely defined linear operator from $\mathcal{D}_R \subset X$ into Y, then we say that R is (T, S)-intertwining if $\mathcal{D}_R$ is invariant under all operators T_a, $a \in \mathcal{A}$ and $RT_a x = S_a Rx$ for all $x \in \mathcal{D}_R$ and $a \in \mathcal{A}$.

Theorem 3 (Żelazko [4])**.** *Let (T, X) be an irreducible representation of an algebra $\mathcal{A}$ on a topological vector space X for which the only continuous (T, T)-intertwining operators are scalar multiples of the identity. Then (T, X) is totally irreducible if and only if for every $k \in \mathbb{N}$ all closed densely defined $(T, T^{(k)})$-intertwining operators are continuous.*

Certainly Theorem 3 is not a very handy criterion for the study of total irreducibility because it requires the investigation of an infinite family of representations and spaces of the corresponding intertwining operators. Nevertheless, it permits to deduce a number of results proved in previous papers of Żelazko and inspires to formulate another version of the problem of Fell and Doran.

As an example of application of this theorem we give a proof of the following result obtained by Żelazko in [3] in case X being an F-space (a completely metrizable topological vector space).

Theorem 4 (Żelazko [3])**.** *Let X be an F-space and let (T, X) be an algebraically irreducible representation of an algebra $\mathcal{A}$ such that $\mathcal{T}' = \mathbb{K}\,\mathrm{id}_X$. Then (T, X) is totally irreducible.*

The proof is immediate. By the algebraic irreduciblity of (T, X) every closed $(T, T^{(k)})$-intertwining operator R is defined on the whole space X and by the closed graph theorem it is continuous. According to Theorem 3 the representation (T, X) is totally irreducible.

Let me finish with a general opinion that the answer to the problem of Fell and Doran is in negative even for the Banach or Hilbert spaces. However nobody was able to find a counterexample.

References

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